ANSWERS MATHEMATICS 1 (FEB11003X) EXAM 21/10/2011

Part I: Basic problems

3 points per problem

Problem 1

Consider the following statement: "If a function f is continuous, then f is differentiable on its domain. " Is this statement correct? And what about the converse statement? Explain your answers.

Final answer:

This statement is false. A continuous function can have a "kink" at which it is not differentiable. For example, $f(x)=|x|$, where the function has a kink at zero and the left–sided limit of the Newton quotient differs from the right-sided limit. However, the reverse statement is true as continuity is a necessary condition for differentiability

Problem 2

Suppose $x^5 + y^4 = 45$, with x and y real variables. Is it possible to express y as a function of x? In addition, is it possible to express x as a function of y? Explain your answers.

Final answer:

We can rewrite the equation as $y=\pm\sqrt[4]{45-x^5}$ or $x=\sqrt[5]{45-y^4}$ so that for each x within its domain, the first expression assigns two values to y, so that there are 2 values of y associated with each value of x, which violates the definition of a function (see page 80). Therefore, y is not a function of x. However, for each y within its domain, the function only assigns one value to x, so that x is a function of y.

Problem 3

Let f be a function that is twice-differentiable on its domain. If the second derivative is zero at a stationary point on the function, is it possible that such stationary point is also a local extreme point? If your answer is "yes", then give an example, otherwise explain why this is not possible.

Final answer:

Yes, an example could be: $y = x^4$. For this function we have $y' = 4x^3$ and $y'' = 12x^2$. Therefore, it has stationary point at $x = 0$ is zero and the second derivative at $x = 0$ is also zero. It can easily be verified by drawing the graph or inspecting the sign diagram of y' around $x = 0$, that the stationary point is in fact is a minimal point (both local and global) of the function.

Problem 4

Let $f(x)$ be some continuous function with domain D. Suppose f is concave for all $x \in D$. Can f also be convex within the domain? Motivate your answer

Final answer:

Yes, it is possible. One example would be the linear function $f(x) = ax + b$, where a and b are constants. As the second derivative of linear function is zero, it is both concave and convex in its domain.

Part II: Multiple choice problems

4 points per problem

Problem 5

Consider the graph of the function $y = f(x)$. If we want to shift the graph c units to the right, then we should replace the function by

A. $y = f(x)$+c B. $y = f(x)$-c C. $y = f(x - c)$ D. $y = f(x + c)$

Final answer:

C

(see page 129 of the book)

Problem 6

The derivative of the function $f(x) = (5x + 2)\left(\frac{3}{x^2} + \frac{1}{x}\right)$ is

A $-17x^{-2} - 12x^{-3}$ B. $-12x^{-2} - 17x^{-3}$

C. $-17x^{-2} + 12x^{-3}$ D. $-15x^{-2} - 13x^{-3}$

Final answer:
A
We first simplify *f(x)* before we compute the derivative: $f(x)=15x^{-1}+5+6x^{-2}+2x^{-1}=17x^{-1}+5+6x^{-2}$ $f'(x)=-17x^{-2}-12x^{-3}$

Problem 7

Calculate the derivative of $f(x)=\left(5\ln x+x^2\right)^2$. The answer is:

A. $2\left(5\ln x+2x^2\right)\left(\frac{5}{x}+2x\right)$ B. $2\left(5\ln x+x^2\right)\left(\frac{5}{x}+2\right)$

C. $2\left(5\ln x+x^2\right)\left(\frac{5}{x}+2x\right)$ D. $2\left(5\ln x+2x\right)\left(\frac{5}{x}+2\right)$

Final answer:
C
$f'(x)=2\left(5\ln x+x^2\right)\left(\frac{5}{x}+2x\right)$

Problem 8

Let $f(x)=7x^2+5x-8$. The interval on which this function is increasing is

A. $[0,-\frac{5}{14}]$ B. $(-\infty,-\frac{5}{14}]$ C. $[-\frac{5}{14},+\infty)$ D. $[-\frac{5}{7},+\infty)$

Final answer:
C
$f'(x)=14x+5$ The function is increasing when $f'(x)\geq 0$, so that $14x+5\geq 0 \Leftrightarrow 14x\geq -5 \Leftrightarrow x\geq -\frac{5}{14} \Leftrightarrow x\in[-\frac{5}{14},+\infty)$

Problem 9

Determine the first derivative of *y* at the point (*x*,*y*)=(1,-2) of $x^3-xy+y^5=-29$ by implicit differentiation. Which of the following answers is correct?

A. $-\frac{5}{19}$ B. $-\frac{5}{39}$ C. $\frac{5}{59}$ D. $-\frac{5}{79}$

Final answer:
D
$y' = -\frac{F_x'}{F_y'} = -\frac{3x^2 - y}{-x + 5y^4}$, by substituting in $(x,y)=(1,-2)$, we have $y' = -\frac{3(1)^2 + 2}{-1 + 5(-2)^4} = -\frac{5}{79}$

Problem 10

Consider the function $F(x, y, z,) = \left(2\sqrt{x} + \frac{5x}{\sqrt{y}} + \frac{y}{\sqrt{x}}\right)^6$. The degree of homogeneity is

A. $\frac{1}{2}$ B. 2 C. 3 D. 4

Final answer:
C
$F(tx, ty, tz) = \left(2\sqrt{tx} + \frac{5tx}{\sqrt{ty}} + \frac{ty}{\sqrt{tx}}\right)^6 = \left(\sqrt{t}(2\sqrt{x} + \frac{5x}{\sqrt{y}} + \frac{y}{\sqrt{x}})\right)^6 = t^3(2\sqrt{x} + \frac{5x}{\sqrt{y}} + \frac{y}{\sqrt{x}})^6$ $= t^3 F(x, y, z)$ Therefore, the degree of homogeneity is 3.

Part III: Calculation problems

5 points per problem

Problem 11

Determine the remainder of the polynomial division $\frac{x^3 - 18x^2 + 11}{x - 3}$.

Final answer:
-124

$(x^3 - 18x^2 + 0x + 11) \div (x-3) = x^2 - 15x - 45$ $\underline{x^3 - 3x^2}$ $\leftarrow x^2(x-3)$ $-15x^2 + 0x$ $\underline{-15x^2 + 45x}$ $\leftarrow -15x(x-3)$ $-45x + 11$ $\underline{-45x + 135}$ $\leftarrow -45(x-3)$ -124 *remainder* Therefore, the remainder of the polynomial division is -124.

Problem 12

Consider the function

$$f(x) = \begin{cases} \dfrac{x^2 + x - 12}{x+4} & \text{if} \quad x \neq -4 \\ c & \text{if} \quad x = -4 \end{cases}$$

What should be the value of c such that f is continuous at -4?

Final answer: c=-7
The function is continuous at -4 when $\lim_{x \to -4} f(x) = f(-4) = c$. As $\lim_{x \to -4} f(x) = \lim_{x \to -4} \dfrac{(x+4)(x-3)}{(x+4)} = -7$, c should be -7

Problem 13

Find the equation for the tangent line to the graph of $f(x) = e^x + 5$ at the point (0,6)

Final answer: $y = x + 6$

First, by differentiating w.r.t. x, we get

$f^{'}(x) = e^{x}$ so that the slope of the tangent line at point (0,6) is $f^{'}(0) = e^{0} = 1$

Therefore, the equation of the tangent line is $y - 6 = 1(x - 0) \Leftrightarrow y = x + 6$

Problem 14

Consider the demand function $D(p)$ of a good that gives demand as function of price p. Suppose that when p=100, demand $D(100)$ is equal to 400 and the price elasticity of demand is equal to -15. What will (approximately) be the level of demand when price p increases to 101?

Final answer:

340

$El_x D(p = 100) = -15$, so when price increases by 1%, i.e. from 100 to 101, demand will decrease by approximately 15%. Because $400 \times 15\% = 60$, it follows that $D(101) \approx 400 - 60 = 340$.

Problem 15

Find the maximum and minimum value of the function $f(x) = 2x^2 + \frac{32}{x^2}$ on the interval [-3, 3].

Final answer:

The correct answer is that the function does not have a maximum value (because $\lim_{x \to o} f(x) = \infty$) and the minimum value is 16 at $x = \pm 2$ (see calculations below).

However, because the question is ill-posed (as the function is not defined for x=0), all answers will be considered correct.

1) Stationary points

$$f' = 4x + \left(-\frac{64}{x^3}\right) = \frac{4x^4 - 64}{x^3} = \frac{4(x^4 - 16)}{x^3} = 0 \Rightarrow x^4 = 16 \Rightarrow x = \pm 2$$

2) Endpoints $x = -3$, $x = 3$

3) $f(-3) = f(3) = 21\frac{5}{9}$, $f(-2) = f(2) = 16$

So the minimum value is 16 at $x = \pm 2$.

Problem 16

Let $f(x, y) = 2x^2 + y^x + e^{3x}$. Find all first- and second-order partial derivatives.

Final answer:

$$f'_x = 4x + y^x \ln y + 3e^{3x}$$
$$f''_{xx} = 4 + y^x (\ln y)^2 + 9e^{3x}$$
$$f'_y = xy^{x-1}$$
$$f''_{yy} = x(x-1)y^{x-2}$$
$$f''_{xy} = f''_{yx} = y^{x-1} + xy^{x-1} \ln y$$

Part IV: Open problems

8 points per problem

Problem 17 (2+2+2+2=8 points)

Consider the function $f(x) = 2x^3 - 8x^2 + 8x + 3$.

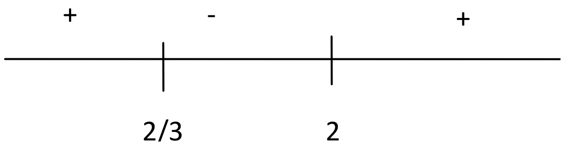
a) Determine the first derivative and the stationary point(s) of f.

$$f' = 6x^2 - 16x + 8 = 2(3x^2 - 8x + 4) = 2(3x - 2)(x - 2)$$
$$f' = 0 \Rightarrow x = 2 \text{ or } \frac{2}{3}$$

b) Classify the stationary point(s) using the first derivative (local minimum, local maximum or not an extreme point).

$f'(0) = 8 > 0$
$f'(1) = 6 - 16 + 8 < 0$
$f'(3) = 54 - 48 + 8 > 0$

So the sign diagram of f' is:



Therefore $x = \frac{2}{3}$ is a local maximum point, $x = 2$ is a local minimum point.

c) Determine the second derivative f'' and all value(s) of x for which $f''(x) = 0$.

$f'' = 12x - 16 = 0 \Rightarrow x = \frac{4}{3}$

d) Determine on which interval(s) the function *f* is concave.

We pick the following 2 points to determine the signs of f''

$f''(0) = -16 < 0$
$f''(2) = 8 > 0$

f is concave when $f'' \le 0$, therefore, *f* is concave on the interval $(-\infty, \frac{4}{3}]$

Problem 18 (2+2+1+3=8 points)

Consider the function $f(x, y) = 2x^3y - 54y - 3x^2 + 10$.

a) Calculate all first order partial derivatives.

$f'_x = 6x^2y - 6x$
$f'_y = 2x^3 - 54$

b) Calculate all second order partial derivatives.

$f''_{xx} = 12xy - 6$
$f''_{yy} = 0$
$f''_{xy} = f''_{yx} = 6x^2$

c) Find all stationary points of *f*.

$f'_x = 0 \Rightarrow x = 0$ or $xy = 1$

First assume $x = 0$, then $f'_y = 0 \Rightarrow 0 - 54 = 0$ which is impossible.

Now assume $x \neq 0$, then $f'_x = 0$ implies $y \neq 0$ so we can take $x = \frac{1}{y}$. Substitution in

$f'_y = 0 \Rightarrow (\frac{1}{y})^3 = 27 \Rightarrow y = \frac{1}{3}, x = 3$

So there is only one stationary point, which is $(3, \frac{1}{3})$.

d) Calculate the value of *f* in every stationary point and use the second-derivative test to classify them (local minimum, local maximum, saddle point or inconclusive).

$f(3, \frac{1}{3}) = -17$

Let $A = f''_{xx}, B = f''_{xy}, C = f''_{yy}$ where

$f''_{xx} = 12xy - 6$
$f''_{yy} = 0$
$f''_{xy} = 6x^2$

Stationary points	A	B	C	$AC - B^2$	Classification
$(3, \frac{1}{3})$	6	54	0	<0	Saddle point

Problem 19 (2+3+3=8 points)

We want to find the local extreme values of $f(x,y)=6xy^2-18y$ subject to $g(x,y)=x^2-y^2=0$.

a) Give the Lagrangian function and the first order conditions of the Lagrange multiplier method for this particular problem.

The Lagrangian function is given by

$$L=6xy^2-18y-\lambda(x^2-y^2)$$

The first order conditions are:

$$L'_x=6y^2-2\lambda x=0 \qquad (1)$$
$$L'_y=12xy-18+2\lambda y=0 \qquad (2)$$
$$x^2-y^2=0 \qquad (3)$$

b) Find the point(s) (x,y) and the corresponding value(s) of λ that satisfy the first order conditions of the Lagrange multiplier method.

From equation (3), we get

$$(x+y)(x-y)=0 \Rightarrow x=\pm y$$

- In case $x=y$, substitute it into (1), we get

$$6y^2=2\lambda y \Rightarrow y=0 \quad \text{or} \quad y=\frac{\lambda}{3}$$

Substitute $y=0$ into (2), we get -18=0, which is impossible.

Substitute $y=\frac{\lambda}{3}$ into (2), we get the following solution

$$(x,y,\lambda)=(1,1,3) \quad \text{or} \quad (-1,-1,-3)$$

- In case $x=-y$, substitute it into (1), we get

$$y(6y+2\lambda)=0 \Rightarrow y=0 \quad \text{or} \quad y=-\frac{\lambda}{3}$$

Substitute $y=0$ into (2), we get -18=0, which is impossible.

Substitute $y=-\frac{\lambda}{3}$ into (2), we get $y^2=-1$, indicating there is no solution.

Together we find 2 points that satisfy the first-order conditions:
$(x, y, \lambda) = (1,1,3)$ or $(-1,-1,-3)$

c) What are the local extreme values and at which point(s) are they attained?

We have

$f''_{11}(x, y) = 0,\ f''_{12}(x, y) = 12y,\ f''_{22}(x, y) = 12x,$
$g'_1(x, y) = 2x, g'_2(x, y) = -2y,$
$g''_{11}(x, y) = 2, g''_{12}(x, y) = 0, g''_{22}(x, y) = -2$

Substituting these functions in

$$D(x, y, \lambda) = (f''_{11} - \lambda g''_{11})(g'_2)^2 - 2(f''_{12} - \lambda g''_{12}) \cdot g'_1 \cdot g'_2 + (f''_{22} - \lambda g''_{22})(g'_1)^2$$

and evaluating this expression in (1,1) and (-1,-1) gives

$D(1,1,3) = 144 > 0$ and $D(-1,-1,-3) = -144 < 0$

Hence, we find that (-1,-1) corresponds to a local maximum, while (1,1) corresponds to a local minimum, with $f(1,1) = -12$ and $f(-1,-1) = 12$.