Erasmus School of Economics

FEB11003X-09 – Mathematics 1

Answers

Personal information student

Name: ________________________________

Student number: ___________________________

Written examination

General information

Date examination:	23 October 2009
Lecturer:	prof. dr. A.P.M. Wagelmans
Time examination:	9:30 until 12:30
Number of questions:	19 questions
Number of pages:	5 pages (incl. cover page)

Instructions ESE

– You are not allowed to use a calculator.
– You are not allowed to use a programmable calculator.
– You are not allowed to use notes (except a cheat sheet, see below).
– You are not allowed to use books.
– You are not allowed to use a dictionary.
– You are allowed to take the examination papers with you.

Additional information

You are allowed to use a cheat sheet, to which the following rules apply:
– Two-sided A4
– Your name and student number in the right upper corner
– Handwritten in your own handwriting
– No photo copies

All material that does not satisfy these rules will be taken away from you and may be considered a fraud attempt.

Use the separate answer sheet to indicate your answers. The exam consists of four parts, each with a different type of problem. For the basic problems, multiple choice problems and calculation problems you will score 3, 4 and 5 points, respectively, per correct answer and no points for incorrect answers. For the open problems (8 points per problem) your score will depend on the answer and the calculation. The exam grade is the result of the formula (10 + number of scored points)/10 , so 63 scored points result in grade 7.3, since (10+63)/10 = 7.3.

Good luck!

Part I: Basic Problems

3 points per problem

Problem 1

Consider the following statement:

For function $y = x^2 - 1$ with domain $\mathbb{R}$, its inverse function is $x = \sqrt{y + 1}$.

Is this statement true or false? Explain why.

> **Solution**
>
> False. Because the function $y = x^2 - 1$ is not one-to-one on its domain $\mathbb{R}$, it has no inverse function. See page 138 of the book.

Problem 2

For a function $y = f(x)$ with domain D, consider the following statement:

For a point a in D, if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$, then $f(x)$ is continuous at the point a.

Is this statement true or false? Explain why.

> **Solution**
>
> False. Because from $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ we only know $\lim_{x \to a} f(x)$ exists. And for f to be continuous at $x = a$, this limit must be equal to $f(a)$. See page 233 of the book.

Problem 3

Suppose we have a function of two variables, $f(x, y)$, and we consider the partial elasticity of f with respect to x. Describe in words the meaning of the partial elasticity of f with respect to x.

> **Solution**
>
> The partial elasticity of f with respect to x measures the percentage (relative) change of the function value, $f(x, y)$, given a one percent increase in the variable x, while y is held constant.

Problem 4

We would like to minimise the convex function $f(x, y) = x^2 - 3xy + 4y^2$, where x and y can take on any real value. Suppose that we have found a stationary point. Is it necessary that we perform a second order test? Explain your answer.

Solution

Because f is a convex function (that is twice continuously differentiable and because the domain, $\mathbb{R}^2$, is a convex set), theorem 13.2.1 tells us that the (only) stationary point is the (global) minimum of the function. Hence, it is not necessary to perform a second-order test.

Part II: Multiple Choice Problems

4 points per problem

Problem 5

The graphs of $f(x)$ and $-f(-x)$ are symmetric about which of the following:

A. the origin B. the x axis C. the y axis D. the line $y = x$

Solution

$f(x)$ and $f(-x)$ are symmetric about the y axis, $f(-x)$ and $-f(-x)$ are symmetric about the x axis, so $f(x)$ and $-f(-x)$ are symmetric about the origin. Answer **A** is correct.

Problem 6

The limit of $\frac{x^3-1}{x-1}$ as $x \to 1$ is:

A. 0 B. 1 C. 3 D. ∞

Solution

$$\lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{x-1} = \lim_{x\to 1}(x^2+x+1) = 3$$

Answer **C** is correct.

Problem 7

Find the elasticity of $y = e^x \ln x$ with respect to x at the point $x = e$.

A. $e + \frac{1}{e}$ B. $e + 1$ C. e D. 1

Solution

$$El_x y = \frac{x}{y} \cdot y' = \frac{x}{e^x \ln x} \cdot (e^x \ln x + e^x \frac{1}{x}) = x + \frac{x}{e^x \ln x} \cdot \frac{e^x}{x} = x + \frac{1}{\ln x}$$

When $x = e$, $El_x y = e + 1/\ln e = e + 1$ Answer (B) is correct.

Problem 8

Compute the degree of homogeneity of the function

$$f(x,y) = \frac{a(xy)^3}{bx^2 + cy^2} \quad ,$$

where a, b and c are constants. Which of the following answers is correct?

A. 1 B. 2 C. 3 D. 4

Solution

$$\begin{aligned} f(tx,ty) &= \frac{a(txty)^3}{b(tx)^2 + c(ty)^2} = \frac{a(t^2xy)^3}{bt^2x^2 + ct^2y^2} = \frac{t^6 a(xy)^3}{t^2\left(bx^2 + cy^2\right)} = t^4 \frac{a(xy)^3}{bx^2 + cy^2} \\ &= t^4 f(x,y) \end{aligned}$$

So the function f is homogeneous of degree 4 (answer **D**).

Problem 9

Consider the function $f(x,y) = x^2 + 2xe^y$ with $x = 2(s-t)$ and $y = \frac{1}{2}ts$.
Compute the partial derivative of $f(x,y)$ w.r.t. t in the point $(t,s) = (1,2)$.

A. $-8 - 2e$ B. -8 C. $4 - 4e$ D. $4 + 2e$

Solution

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt} + \frac{df}{dy}\frac{dy}{dt} = (2x + 2e^y)(-2) + 2xe^y \frac{1}{2}s = -4x - 4e^y + xse^y$$

We fill in the point $(t,s) = (1,2)$ to get $x = 2(2-1) = 2$ and $y = \frac{1}{2} \cdot 1 \cdot 2 = 1$, so

$$\frac{df}{dt} = -4 \cdot 2 - 4e^1 + 2 \cdot 2 \cdot e^1 = -8 - 4e + 4e = -8 \quad .$$

Answer **B** is correct.

Problem 10

Let $f(x,y) = yx^y e^{2x+y}$. Compute the partial elasticity of f w.r.t. x in the point $(x,y) = (3,2)$. Which of the following answers is correct?

A. 4 B. 7 C. 8 D. 20

Solution

$f_x'(x,y) = y\left(yx^{y-1}e^{2x+y} + x^y e^{2x+y}\cdot 2\right)$

$El_x(x,y) = \dfrac{xf_x'(x,y)}{f(x,y)} = \dfrac{xy\left(yx^{y-1}e^{2x+y} + 2x^y e^{2x+y}\right)}{yx^y e^{2x+y}} = \dfrac{x^y e^{2x+y}(y+2x)}{x^y e^{2x+y}} = y + 2x$

$El_x(3,2) = 2 + 2\cdot 3 = 8$, so the correct answer is **C**.

Part III: Calculation Problems

5 points per problem

Problem 11

Solve the following equation for x

$$3^{2x^2-5x+6} = \frac{1}{9}\left(3^x\right)^{x+1}$$

Solution

$$\begin{aligned}
3^{2x^2-5x+6} &= \frac{1}{9}\left(3^x\right)^{x+1} \\
\Leftrightarrow 3^{2x^2-5x+6} &= \frac{3^{x^2+x}}{3^2} \\
\Leftrightarrow 3^{2x^2-5x+6} &= 3^{x^2+x-2} \\
\Leftrightarrow 2x^2-5x+6 &= x^2+x-2 \\
\Leftrightarrow x^2-6x+8 &= 0 \\
\Leftrightarrow (x-2)(x-4) &= 0 \\
\Leftrightarrow x = 2 \text{ or } x = 4 &
\end{aligned}$$

Problem 12

Find the value(s) of k which make(s) the remainder of $(k^2x^2 + kx - 6) \div (x-2)$ equal to zero.

Solution

$$
\begin{array}{l}
(k^2x^2+kx-6)\div(x-2) = k^2x+k+2k^2 \\
\underline{k^2x^2-2k^2x} \\
\quad (k+2k^2)x-6 \\
\quad \underline{(k+2k^2)x-2(k+2k^2)} \\
\qquad\qquad -6+2(k+2k^2)
\end{array}
$$

The remainder equals zero, so

$$
\begin{aligned}
-6+2(k+2k^2) &= 0 \\
\Leftrightarrow -3+k+2k^2 &= 0 \\
\Leftrightarrow (2k+3)(k-1) &= 0 \\
\Leftrightarrow k &= -3/2 \text{ or } 1
\end{aligned}
$$

Problem 13

Consider the function $y = e^{\frac{1}{3}x-2}+3$ with domain $[6,\infty)$. What are the inverse and its domain? If the inverse does not exist, report 'no inverse'.

Solution

$$
\begin{aligned}
e^{\frac{1}{3}x-2}+3 &= y \\
\Leftrightarrow e^{\frac{1}{3}x-2} &= y-3 \\
\Leftrightarrow \frac{1}{3}x-2 &= \ln(y-3) \\
\Leftrightarrow \frac{1}{3}x &= \ln(y-3)+2 \\
\Leftrightarrow x &= 3\ln(y-3)+6
\end{aligned}
$$

So the inverse function is $x = 3\ln(y-3)+6$. The domain of the inverse is the range of the original function.

$$f(6) = e^{\frac{1}{3}6-2}+3 = e^0+3 = 1+3 = 4$$

$$\lim_{x\to\infty} e^{\frac{1}{3}x-2}+3 = \infty$$

So the domain of the inverse is $[4,\infty)$.

Problem 14

Find the slope of the tangent line to the curve $(xy+1)^2 + x^2 = 5$ at $(x, y) = (1, 1)$.

Solution

The slope of the tangent line is y'. Using implicit differenciation on the curve, we get

$$\begin{aligned} 2(xy+1)(y+xy') + 2x &= 0 \\ \Rightarrow (xy+1)(y+xy') + x &= 0 \\ \Rightarrow (xy+1)y + (xy+1)xy' + x &= 0 \\ \Rightarrow y' &= \frac{-x-(xy+1)y}{(xy+1)x} \end{aligned}$$

At $(x, y) = (1, 1)$, $y' = \frac{-1-2}{2} = -\frac{3}{2}$.

Problem 15

For the function $f(x) = \frac{x^2+4}{x}$ on the interval $[1, 4]$, find the maximum and minimum values of $f(x)$ and the corresponding x values.

Solution

Step 1: find the stationary point(s) of $f(x) = \frac{x^2+4}{x} = x + \frac{4}{x}$

$$f'(x) = 1 - \frac{4}{x^2} = 0 \Leftrightarrow x^2 = 4 \qquad \Leftrightarrow x = \pm 2$$

Note that -2 is not in the interval $[1, 4]$, so we find one candidate point $x = 2$, $f(2) = \frac{4+4}{2} = 4$.
Step 2: compute the function value at the boundary points.

$$\begin{aligned} f(1) &= \frac{1+4}{1} = 5 \\ f(4) &= \frac{16+4}{4} = 5 \end{aligned}$$

Step 3: compare the function values at the above candidate points. We conclude that, on the given interval, the maximum value of $f(x)$ is 5 at $x = 1$ and 4, the minimum value is 4 at $x = 2$.

Problem 16

Let $f(x,y) = x^y + \ln(3y^2+1) + 2xy$.
Find all first- and second-order partial derivatives.

Solution

$$
\begin{aligned}
f'_x(x,y) &= yx^{y-1} + 2y \\
f'_y(x,y) &= x^y \ln(x) + \frac{6y}{3y^2+1} + 2x \\
f''_{xx}(x,y) &= y(y-1)x^{y-2} \\
f''_{xy}(x,y) &= x^{y-1} + yx^{y-1}\ln(x) + 2 = f''_{yx}(x,y) \\
f''_{yy}(x,y) &= x^y \ln(x)\ln(x) + \frac{6(3y^2+1) - 6y \cdot 6y}{(3y^2+1)^2} = x^y\,(\ln(x))^2 + \frac{6-18y^2}{(3y^2+1)^2}
\end{aligned}
$$

Part IV: Open problems

8 points per problem

Problem 17 ($2+2+2+2$ points)

Consider the function $f(x) = \frac{1}{5}x^5 - 3x^3$.

a. Calculate the first derivative and the stationary point(s).

b. Classify the stationary point(s) using only the first derivative (local minima, local maxima or not a local extreme point).

c. Calculate the second derivative $f''(x)$ and the point(s) for which $f''(x) = 0$.

d. Classify the point(s) from part c using the second derivative (inflection point or not an inflection point).

(It might be useful to know that $\sqrt{2} \approx 1.41$, $\sqrt{3} \approx 1.73$, $\sqrt{5} \approx 2.24$, $\sqrt{6} \approx 2.45$, $\sqrt{7} \approx 2.65$.)

Solution

a. $f'(x) = x^4 - 9x^2 = x^2(x^2 - 9)$
$f'(x) = 0 \Leftrightarrow x^2(x^2-9) = x^2(x+3)(x-3) = 0 \Leftrightarrow x = 0$ or ± 3.
So the stationary points are $0, -3, 3$.

b. $f'(-4) = (-4)^2((-4)^2 - 9) = 4^2(4^2 - 9) > 0$
$f'(-1) = (-1)^2((-1)^2 - 9) = 1 - 9 < 0$
$f'(1) = 1^2(1^2 - 9) = 1 - 9 < 0$
$f'(4) = 4^2(4^2 - 9) > 0$
So $f'(x) > 0$ on $(-\infty, -3)$, $f'(x) < 0$ on $(-3, 0)$ and $(0, 3)$, $f'(x) > 0$ on $(-3, \infty)$.
Hence, $x = -3$ is a local maximum point, $x = 3$ is a local minimum point, and $x = 0$ is not an extreme point.

c. $f''(x) = 4x^3 - 18x = 2x(2x^2 - 9)$
$f''(x) = 0 \Leftrightarrow x = 0 \text{ or } 2x^2 - 9 = 0 \Leftrightarrow x = 0 \text{ or } x^2 = 9/2 \Leftrightarrow x = 0 \text{ or } \pm 3/\sqrt{2}$

d. $f''(-3) = 2 \cdot (-3)(2 \cdot (-3)^2 - 9) = -6(18 - 9) < 0$
$f''(-1) = 2 \cdot (-1)(2 \cdot (-1)^2 - 9) = -2(2 - 9) > 0$
$f''(1) = 2 \cdot 1(2 \cdot 1^2 - 9) = 2(2 - 9) < 0$
$f''(3) = 2 \cdot 3(2 \cdot 3^2 - 9) = 6(18 - 9) > 0$
So $f''(x) < 0$ on $(-\infty, -3/\sqrt{2})$, $f''(x) > 0$ on $(-3/\sqrt{2}, 0)$, $f''(x) < 0$ on $(0, 3/\sqrt{2})$, $f''(x) > 0$ on $(3/\sqrt{2}, \infty)$.
Hence, $x = 0$, $\pm 3/\sqrt{2}$ are all inflection points.

Problem 18 ($3+3+2$ points)

Consider the function $f(x,y) = x^3 + (x-y)^3 - 15x + 3y$.

a. Calculate the first and second order partial derivatives of f.

b. Calculate the stationary points of f. (It is not necessary to compute the function value of f in every stationary point.)

c. Use the second order test to classify the stationary points (i.e. local minimum, local maximum, saddle point or inconclusive).

Solution

a.

$$\begin{aligned}
f'_x(x,y) &= 3x^2 + 3(x-y)^2 - 15 \\
f'_y(x,y) &= -3(x-y)^2 + 3 \\
f''_{xx}(x,y) &= 6x + 6(x-y) \\
f''_{xy} &= -6(x-y) = f''_{yx} \\
f''_{yy} &= 6(x-y)
\end{aligned}$$

b. In the stationary points it holds that $f'_x(x,y) = 0$ and $f'_y(x,y) = 0$.
$f'_y(x,y) = -3(x-y)^2 + 3 = 0 \Leftrightarrow (x-y)^2 = 1 \Leftrightarrow x - y = 1$ or $x - y = -1 \Leftrightarrow y = x - 1$ or $y = x + 1$.

Suppose $y = x - 1$. Substituting this in $f'_x(x,y) = 3x^2 + 3(x-y)^2 - 15 = 0$ gives $3x^2 + 3\left(x - x + 1\right)^2 - 15 = 0 \Leftrightarrow 3x^2 - 12 = 0 \Leftrightarrow x^2 = 4 \Leftrightarrow x = \pm 2$. If $x = -2$, then $y = -2 - 1 = -3$. If $x = 2$, then $y = 2 - 1 = 1$.

Suppose $y = x + 1$. Substituting this in $f'_x(x,y) = 3x^2 + 3(x-y)^2 - 15 = 0$ gives $3x^2 + 3\left(x - x - 1\right)^2 - 15 = 0 \Leftrightarrow 3x^2 - 12 = 0 \Leftrightarrow x = \pm 2$. If $x = -2$, then $y = -2 + 1 = -1$. If $x = 2$, then $y = 2 + 1 = 3$.

The stationary points are $(-2,-3)$, $(2,1)$, $(-2,-1)$ and $(2,3)$.

c. Using the second order test we get the following table.

(x,y)	A	B	C	$AC - B^2$	type
$(-2,-3)$	-6	-6	6	$-2 \cdot 6^2$	saddle point
$(-2,-1)$	-18	6	-6	$18 \cdot 6 - 6^2 > 0$	local maximum
$(2,1)$	18	-6	6	$18 \cdot 6 - 6^2 > 0$	local minimum
$(2,3)$	6	6	-6	$-2 \cdot 6^2$	saddle point

Problem 19 ($3+3+2$ points)

We would like to find the maximum of

$$f(x,y) = x^2 + x^2y \quad \text{subject to} \quad g(x,y) = 2x^2 + y = 3 \quad .$$

It is given that this maximisation problem has a solution.

a. Give the Lagrangian function and the first order conditions of the Lagrange multiplier method for this problem.

b. Find the point(s) (x,y) and the corresponding value(s) of λ that satisfy the first order conditions of the Lagrange multiplier method.

c. What is the maximum value and at which point(s) is it attained?

Solution

a. The Lagrangian function in this case is

$$\mathcal{L}(x,y) = f(x,y) - \lambda(g(x,y) - c) = x^2 + x^2y - \lambda\left(2x^2 + y - 3\right) \quad .$$

In general, the first order conditions are:

$$\mathcal{L}'_x(x,y) = f'_x(x,y) - \lambda g'_x(x,y) = 0$$
$$\mathcal{L}'_y(x,y) = f'_y(x,y) - \lambda g'_y(x,y) = 0$$
$$g(x,y) = c \quad .$$

For this specific problem it gives:

$$\mathcal{L}'_x(x,y) = 2x + 2xy - 4x\lambda = 0 \qquad (1)$$
$$\mathcal{L}'_y(x,y) = x^2 - \lambda = 0 \qquad (2)$$
$$2x^2 + y - 3 = 0 \quad . \qquad (3)$$

b. From (2) we get $x^2 - \lambda = 0 \Leftrightarrow \lambda = x^2$.
From (3) we get $2x^2 + y - 3 = 0 \Leftrightarrow y = 3 - 2x^2$.
If we substitute these expressions for y and λ in (1), then we get

$$\begin{aligned} 2x + 2x(3-2x^2) - 4x(x^2) &= 0 \\ \Leftrightarrow 2x + 6x - 4x^3 - 4x^3 &= 0 \\ \Leftrightarrow -8x^3 + 8x &= 0 \\ \Leftrightarrow x(-8x^2 + 8) &= 0 \\ \Leftrightarrow x = 0 \quad &\text{or} \quad 8x^2 = 8 \\ \Leftrightarrow x = 0 \quad &\text{or} \quad x = \pm 1 \quad . \end{aligned}$$

If $x = 0$, then $y = 3 - 2 \cdot 0^2 = 3$ and $\lambda = 0^2 = 0$.
If $x = 1$, then $y = 3 - 2 \cdot 1^2 = 1$ and $\lambda = 1^2 = 1$.
If $x = -1$, then $y = 3 - 2(-1)^2 = 1$ and $\lambda = (-1)^2 = 1$.
Hence, the points that satisfy the first order conditions are $(0,3)$ with $\lambda = 0$, $(1,1)$ with $\lambda = 1$ and $(-1,1)$ with $\lambda = 1$.

c. It is given that the solution of the maximisation problem exists, so we only need to compare the function values in each of the three candidate points found in part b.
$f(0,3) = 0^2 + 0^2 \cdot 3 = 0 \qquad f(-1,1) = (-1)^2 + (-1)^2 1 = 2$
$f(1,1) = 1^2 + 1^2 \cdot 1 = 2.$
We conclude that the maximum value is 2, attained at the points $(-1,1)$ and $(1,1)$.