Erasmus School of Economics

FEB11003X-09 – Mathematics 1

Version with answers

Personal information student

Name: ______________________________

Student number: ________________________

Written examination Resit

General information

Date examination:	12 July 2010
Lecturer:	prof. dr. A.P.M. Wagelmans
Time examination:	9:30 until 12:30
Number of questions:	19 questions
Number of pages:	5 pages (incl. cover page)

Instructions ESE

– You are not allowed to use a calculator.
– You are not allowed to use a programmable calculator.
– You are not allowed to use notes (except a cheat sheet, see below).
– You are not allowed to use books.
– You are not allowed to use a dictionary.
– You are allowed to take the examination papers with you.

Additional information

You are allowed to use a cheat sheet, to which the following rules apply:
– Two-sided A4
– Your name and student number in the right upper corner

All material that does not satisfy these rules will be taken away from you and may be considered a fraud attempt.

Use the separate answer sheet to indicate your answers. The exam consists of four parts, each with a different type of problem. For the basic problems, multiple choice problems and calculation problems you will score 3, 4 and 5 points, respectively, per correct answer and no points for incorrect answers. For the open problems (8 points per problem) your score will depend on the answer and the calculation. The exam grade is the result of the formula (10 + number of scored points)/10 , so 63 scored points result in grade 7.3, since (10+63)/10 = 7.3.

Good luck!

Part I: Basic Problems
3 points per problem

Problem 1

Suppose $y^2 = x^2 + 1$, can we say y is a function of x? Explain your answer.

Solution

No. Because a function is a rule which to each element in a set A assigns one and only one element in a set B (textbook page 151 (1)). While here there are 2 values of y are associated with each value of x, $y = \pm\sqrt{x^2+1}$.

Problem 2

Suppose the price elasticity of demand for sugar is -0.8. Explain in words how you would interpret this elasticity.

Solution

This means that an increase of 1% (or 10%) in the price of sugar would lead to a decrease of 0.8% (or 8%) in the demand.

Problem 3

Consider the function $f(x,y) = x^2 - xy + y^2$. Is this function concave, convex, or neither in its domain $\mathbb{R}^2$?

Solution

$f'_x = 2x - y$, $f'_y = -x + 2y$, $f''_{xx} = 2 > 0$, $f''_{yy} = 2 > 0$, $f''_{xy} = -1$, then $f''_{xx}f''_{yy} - (f''_{xy})^2 = 4 - (-1)^2 = 3 > 0$. So $f(x,y)$ is convex in its domain. (see textbook page 459, note 2 of theorem 13.2.1)

Problem 4

Suppose we use the Lagrange Multiplier method to optimize $f(x,y)$ subject to $g(x,y) = c$. We get the solution point (x^*, y^*) and the optimal value f^*. We also obtain a value for λ and it holds that $\lambda = df^*/dc$. Interpret λ.

Solution

λ reflects how sensitive the optimal value f^* is for the changes in c.

Part II: Multiple Choice Problems
4 points per problem

Problem 5

Suppose $f(x) = f(-x)$ for all x in its domain, then $f(x)$ is:

A. ≥ 0 B. ≤ 0 C. symmetric about the x axis D. symmetric about the y axis

Solution

D .
$f(x) = f(-x)$ means the graph of $y = f(x)$ remains unchanged after reflecting about the y-axis. So $f(x)$ is symmetric about the y axis.

Problem 6

$\lim_{x\to -1} \frac{x^2-3x-4}{x+1} =$

A. -5 B. -2 C. 1 D. ∞

Solution

A.

$$\lim_{x\to -1} \frac{x^2-3x-4}{x+1} = \lim_{x\to -1} \frac{(x+1)(x-4)}{x+1} = \lim_{x\to -1} (x-4) = -1-4 = -5$$

Problem 7

For which values of x is the function $f(x) = -x^2e^x$ increasing?

A. $x \geq 0$ B. $x \leq -2$ C. $x \leq -2$ and $x \geq 0$ D. $-2 \leq x \leq 0$

Solution

D.
This problem is equivalent to finding the x values that satisfy $f'(x) \geq 0$.

$$f'(x) = -\left(2xe^x + x^2e^x\right) = -x(x+2)e^x \geq 0$$
$$\Leftrightarrow x(x+2) \leq 0 \Leftrightarrow -2 \leq x \leq 0$$

Problem 8

Compute the degree of homogeneity of the function $f(x,y) = \left(\sqrt{x} + \frac{ax}{\sqrt{y}}\right)^4$, where a is a constant. Which of the following answers is correct?

A. 1 B. 2 C. 3 D. 4

Solution

B.

$$f(tx,ty) = \left(\sqrt{tx} + \frac{atx}{\sqrt{ty}}\right)^4 = \left(\sqrt{t}\sqrt{x} + \frac{a\sqrt{tx}}{\sqrt{y}}\right)^4 = \left(\sqrt{t}\right)^4 \left(\sqrt{x} + \frac{ax}{\sqrt{y}}\right)^4 = t^2 f(x,y)$$

Problem 9

Compute the slope of the tangent line to the curve $x^3 - xy^2 + y^3 = 2$ in the point $(x,y) = (1,2)$.Which of the following answers is correct?

A. $\frac{1}{8}$ B. 11 C. $\frac{1}{4}$ D. $-\frac{5}{12}$

Solution

A.
The slope of the tangent line to the curve at (1,2) is $y'(1,2)$. Implicit differenciation on the eqation yields

$$\begin{aligned} 3x^2 - (y^2 + x \cdot 2y \cdot y') + 3y^2 \cdot y' &= 0 \\ \Leftrightarrow 3x^2 - y^2 - 2xy \cdot y' + 3y^2 y &= 0 \\ \Leftrightarrow 3x^2 - y^2 &= (2xy - 3y^2)y' \\ \Leftrightarrow y' &= \frac{3x^2 - y^2}{2xy - 3y^2} \end{aligned}$$

So $y'(1,2) = \dfrac{3-4}{2\cdot 2 - 3\cdot 4} = \dfrac{-1}{-8} = \dfrac{1}{8}$

Problem 10

Let $f(x,y) = 2y^x e^{x-y}$. Compute the partial elasticity of f w.r.t. y . Which of the following answers is correct?

A. $x + y$ B. $x - y$ C. $xy - 1$ D. $xy + 1$

Solution

B.

$$f'_y(x,y) = 2(xy^{x-1}e^{x-y} + y^x \cdot e^{x-y} \cdot (-1)) = 2e^{x-y}(xy^{x-1} - y^x)$$

$$\text{El}_y f(x,y) = f'_y(x,y)\frac{y}{f(x,y)} = 2e^{x-y}(xy^{x-1} - y^x) \cdot \frac{y}{2y^x e^{x-y}} = \frac{xy^x - y^x y}{y^x} = x - y$$

Part III: Calculation Problems
5 points per problem

Problem 11

Solve the following equation for x

$$\frac{1}{x+5} - \frac{1}{x^2+3x-10} = \frac{1}{2-x}$$

Solution

$x = -1$.

$$\frac{1}{x+5} - \frac{1}{x^2+3x-10} = \frac{1}{2-x}$$
$$\Leftrightarrow \frac{1}{x+5} - \frac{1}{(x-2)(x+5)} = \frac{1}{2-x}$$
$$\Leftrightarrow \frac{(x-2)-1}{(x-2)(x+5)} = -\frac{(x+5)}{(x-2)(x+5)}$$
$$\Leftrightarrow x - 3 = -x - 5$$
$$\Leftrightarrow x = -1$$

Problem 12

Find the value(s) of k which make(s) the remainder of $(x^3 - ax) \div (x + k)$ equal to zero. Here a is a positive constant.

Solution

$k = 0$ and $k = \pm\sqrt{a}$.

$$(x^3 \quad - ax \quad) \div (x + k) = x^2 - kx + (-a + k^2)$$
$$\underline{x^3 + kx^2}$$
$$-kx^2 - ax$$
$$\underline{-kx^2 - k^2 x}$$
$$(-a + k^2)x$$
$$\underline{(-a + k^2)x + (-a + k^2)k}$$
$$-(-a + k^2)k$$

The remainder $-(-a + k^2)k = 0 \Leftrightarrow (k^2 - a)k = 0 \Leftrightarrow k = \pm\sqrt{a}, k = 0$

Problem 13

Consider the function $y = 2\ln(x + 3) - 6$ with domain $[-2, \infty)$. What are the inverse and its domain? If the inverse does not exist, report 'no inverse'.

Solution

The inverse function is $x = e^{\frac{1}{2}y+3} - 3$. The domain of the inverse is $[-6, \infty)$.

$$y = 2\ln(x + 3) - 6 \Leftrightarrow \frac{y + 6}{2} = \ln(x + 3) \Leftrightarrow e^{\frac{y}{2}+3} = x + 3 \Leftrightarrow x = e^{\frac{1}{2}y+3} - 3$$

Then we need to find the range of the original function. $\ln(x+3)$ is an increasing function, so $y \geq 2\ln(-2 + 3) - 6 = -6$.

Problem 14

Consider the function $f(x, y) = e^{x^2} + ye^{xy}$ with $x = 2t - s$, $y = ts$. Compute the partial derivative of $f(x, y)$ w.r.t. t in the point $(t, s) = (1, 2)$.

Solution

10 .

$$\frac{\partial f}{\partial x} = e^{x^2} 2x + ye^{xy} \cdot y, \ \frac{\partial f}{\partial y} = e^{xy} + y \cdot e^{xy} \cdot x$$

$$\frac{\partial x}{\partial t} = 2, \ \frac{\partial y}{\partial t} = s$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = 2(e^{x^2} 2x + ye^{xy}) + s(e^{xy} + y \cdot e^{xy} \cdot x)$$

We fill in $(t, s) = (1, 2)$ to get $x(1, 2) = 0$ and $y(1, 2) = 2$. Then we plug all these values of x, y, t, s into the above formula, we get

$$\frac{\partial f}{\partial t} = 2 \cdot 4 + 2 \cdot (1 + 0) = 10$$

Problem 15

For the function $f(x) = \frac{32}{x} + 2x$ on the interval [2, 16], find the maximum and minimum values of $f(x)$ and the corresponding x values.

Solution

The maximum of $f(x)$ is 34 at $x = 16$, the minimum of $f(x)$ is 16 at $x = 4$.
Step 1: find the stationary point(s) of $f(x) = \frac{32}{x} + 2x$

$$f'(x) = -\frac{32}{x^2} + 2 = 0 \Leftrightarrow x^2 = 16 \qquad \Leftrightarrow x = \pm 4$$

Note that -4 is not in the interval [2, 16], so we find one candidate point $x = 4$, $f(4) = 8 + 8 = 16$.
Step 2: compute the function value at the boundary points.

$$f(2) = 16 + 4 = 20$$
$$f(16) = 2 + 32 = 34$$

Step 3: compare the function values at the above candidate points. We conclude that, on the given interval, the maximum value of $f(x)$ is 34 at $x = 16$, the minimum value is 16 at $x = 4$.

Problem 16

Let $f(x,y) = y^x + \ln xy^2$.
Find all first- and second-order partial derivatives.

Solution

$$
\begin{aligned}
f'_x(x,y) &= y^x \ln y + \frac{y^2}{xy^2} = y^x \ln y + \frac{1}{x} \\
f'_y(x,y) &= xy^{x-1} + \frac{2}{y} \\
f''_{xx}(x,y) &= y^x (\ln y)^2 - \frac{1}{x^2} \\
f''_{xy}(x,y) &= xy^{x-1} \ln y + y^x \frac{1}{y} = y^{x-1}(x \ln y + 1) = f''_{yx}(x,y) \\
f''_{yy}(x,y) &= x(x-1)y^{x-2} - \frac{2}{y^2}
\end{aligned}
$$

Part IV: Open problems
8 points per problem

Problem 17 ($2 + 2 + 2 + 2$ **points)**

Consider the function $f(x) = \frac{1}{x^2+2}$.

a. Calculate the first derivative and the stationary point(s).

b. Classify the stationary point(s) using only the first derivative (local minima, local maxima or not a local extreme point).

c. Calculate the second derivative $f''(x)$ and the point(s) for which $f^{''}(x) = 0$.

d. Classify the point(s) from part c using the second derivative (inflection point or not an inflection point).

   (It might be useful to know that $\sqrt{2} \approx 1.41$, $\sqrt{3} \approx 1.73$, $\sqrt{5} \approx 2.24$, $\sqrt{6} \approx 2.45$, $\sqrt{7} \approx 2.65$.)

Solution

a. $f'(x) = (-1) \cdot (x^2+2) \cdot 2x = \frac{-2x}{(x^2+2)^2}$
$f'(x) = 0 \Leftrightarrow x = 0.$
So the stationary point is $x = 0$.

b. $f'(x) > 0$ on $(-\infty, 0)$, $f'(x) < 0$ on $(0, \infty)$.
Hence, $x = 0$ is a local maximum point.

c.

$$\begin{aligned} f''(x) &= \frac{(-2) \cdot (x^2+2)^2 - (-2x) \cdot 2(x^2+2) \cdot 2x}{(x^2+2)^4} \\ &= \frac{(-2x^2-4)(x^2+2) + 8x^2(x^2+2)}{(x^2+2)^4} \\ &= \frac{(6x^2-4)(x^2+2)}{(x^2+2)^4} \end{aligned}$$

$f''(x) = 0 \Leftrightarrow 6x^2 - 4 = 0 \Leftrightarrow$ x= $\pm\sqrt{\frac{2}{3}} = \pm\frac{\sqrt{6}}{3}$

d. $f''(-1) > 0$
$f''(0) < 0$
$f''(1) > 0$
So $f''(x) > 0$ on $(-\infty, -\frac{\sqrt{6}}{3})$, $f''(x) < 0$ on $(-\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{3})$, $f''(x) > 0$ on $(\frac{\sqrt{6}}{3}, \infty)$.
Hence, $x = \pm\frac{\sqrt{6}}{3}$ are both inflection points.

Problem 18 ($3+3+2$ **points)**

Consider the function $f(x,y) = x^2y^2 - y^2 - 4x^2 + 9$.

a. Calculate the first and second order partial derivatives of f.

b. Calculate the stationary points of f. (It is not necessary to compute the function value of f in every stationary point.)

c. Use the second order test to classify the stationary points (i.e. local minimum, local maximum, saddle point or inconclusive).

Solution

a.

$$\begin{aligned} f'_x(x,y) &= 2xy^2 - 8x \\ f'_y(x,y) &= 2yx^2 - 2y \\ f''_{xx}(x,y) &= 2y^2 - 8 \\ f''_{xy} &= 4xy \\ f''_{yy} &= 2x^2 - 2 \end{aligned}$$

b. In the stationary points it holds that $f'_x(x,y) = 0$ and $f'_y(x,y) = 0$.
$f'_x(x,y) = 2xy^2 - 8x = 2x(y^2 - 4) = 0 \Leftrightarrow x = 0$ or $y^2 = 4 \Leftrightarrow x = 0$ or $y = \pm 2$.

Suppose $x = 0$. Substituting this in $f'_y(x,y) = 2y(x^2 - 1) = 0$ gives $-2y = 0 \Leftrightarrow y = 0$.

Suppose $y = 2$. Substituting this in $f'_y(x,y) = 2y(x^2 - 1) = 0$ gives $4(x^2 - 1) = 0 \Leftrightarrow x^2 - 1 = 0 \Leftrightarrow x = \pm 1$. Suppose $y = -2$. We also get $x = \pm 1$.

So the stationary points are $(0,0)$, $(1,2)$, $(-1,2)$, $(1,-2)$, and $(-1,-2)$.

c. Using the second order test we get the following table.

(x,y)	f''_{xx}	f''_{xy}	f''_{yy}	$f''_{xx}f''_{yy} - (f''_{xy})^2$	type
(0,0)	-8	0	-2	$16 > 0$	local maximum
$(1,2)$	0	8	0	$-8^2 < 0$	saddle point
$(-1,2)$	0	-8	0	$-8^2 < 0$	saddle point
$(1,-2)$	0	-8	0	$-8^2 < 0$	saddle point
$(-1,-2)$	0	8	0	$-8^2 < 0$	saddle point

Problem 19 ($3 + 3 + 2$ points)

Find the maximum and minimum values of

$$f(x,y) = 5xy \quad \text{subject to} \quad g(x,y) = x^2 + y^2 = 18 \quad .$$

a. Give the Lagrangian function and the first order conditions of the Lagrange multiplier method for this problem.

b. Find the point(s) (x,y) and the corresponding value(s) of λ that satisfy the first order conditions of the Lagrange multiplier method.

c. What are the maximum and minimum values of $f(x,y)$ and at which point(s) are they attained? Note that $f(x,y)$ is continuous and the constraint curve is a closed bounded set (a circle).

Solution

a. The Lagrangian function in this case is

$$\mathcal{L}(x,y) = f(x,y) - \lambda(g(x,y) - c) = 5xy - \lambda\left(x^2 + y^2 - 18\right) \quad .$$

The first order conditions are:

$$\mathcal{L}'_x(x,y) = 5y - 2x\lambda = 0 \quad (1)$$
$$\mathcal{L}'_y(x,y) = 5x - 2y\lambda = 0 \quad (2)$$
$$x^2 + y^2 = 18 \quad (3)$$

b. $(x,y) = (0,0)$ does not satisfy the constrain, so from (1) we get $\lambda = 5y/2x$. From (2) we get $\lambda = 5x/2y$.

Then we have $5y/2x = 5x/2y \Leftrightarrow 10x^2 = 10y^2 \Leftrightarrow x^2 = y^2$. Instert this expression in (3), it yields $2y^2 = 18 \Leftrightarrow y = \pm 3$. So $x^2 = y^2 = 9 \Leftrightarrow x = \pm 3$. We can calculate the corresponding value of λ from $\lambda = 5x/2y$.
Hence, the points that satisfy the first order conditions are $(3,3)$ with $\lambda = 5/2$, $(3,-3)$ with $\lambda = -5/2$, $(-3,3)$ with $\lambda = -5/2$, and $(-3,-3)$ with $\lambda = 5/2$.

c. Because $f(x,y)$ is continuous and the constraint curve is a closed bounded set, the extreme value theorem (see textbook page 475) ensures that solutions exist. The solution candicates are (3, 3), (3, -3), (-3, 3), (-3, -3).
Here $f(3,3) = f(-3,-3) = 45$, $f(3,-3) = f(-3,3) = -45$.
So we conclude that the maximum value of $f(x,y)$ is 45, attained at the points $(3,3)$ and $(-3,-3)$; the minimum value of $f(x,y)$ is -45, attained at the points $(3,-3)$ and $(-3,3)$.