

ANSWERS MATHEMATICS 1 (FEB11003X) EXAM 21/10/2011

Part I: Basic problems

3 points per problem

Problem 1

Consider the following statement: "If a function f is continuous, then f is differentiable on its domain." Is this statement correct? And what about the converse statement? Explain your answers.

Final answer:

This statement is false. A continuous function can have a "kink" at which it is not differentiable. For example, $f(x) = |x|$, where the function has a kink at zero and the left-sided limit of the Newton quotient differs from the right-sided limit. However, the reverse statement is true as continuity is a necessary condition for differentiability

Problem 2

Suppose $x^5 + y^4 = 45$, with x and y real variables. Is it possible to express y as a function of x ? In addition, is it possible to express x as a function of y ? Explain your answers.

Final answer:

We can rewrite the equation as $y = \pm\sqrt[4]{45 - x^5}$ or $x = \sqrt[5]{45 - y^4}$ so that for each x within its domain, the first expression assigns two values to y , so that there are 2 values of y associated with each value of x , which violates the definition of a function (see page 80). Therefore, y is not a function of x . However, for each y within its domain, the function only assigns one value to x , so that x is a function of y .

Problem 3

Let f be a function that is twice-differentiable on its domain. If the second derivative is zero at a stationary point on the function, is it possible that such stationary point is also a local extreme point? If your answer is "yes", then give an example, otherwise explain why this is not possible.

Final answer:

Yes, an example could be: $y = x^4$. For this function we have $y' = 4x^3$ and $y'' = 12x^2$. Therefore, it has stationary point at $x = 0$ is zero and the second derivative at $x = 0$ is also zero. It can easily be verified by drawing the graph or inspecting the sign diagram of y' around $x = 0$, that the stationary point is in fact is a minimal point (both local and global) of the function.

Problem 4

Let $f(x)$ be some continuous function with domain D . Suppose f is concave for all $x \in D$. Can f also be convex within the domain? Motivate your answer

Final answer:

Yes, it is possible. One example would be the linear function $f(x) = ax + b$, where a and b are constants. As the second derivative of linear function is zero, it is both concave and convex in its domain.

Part II: Multiple choice problems

4 points per problem

Problem 5

Consider the graph of the function $y = f(x)$. If we want to shift the graph c units to the right, then we should replace the function by

- A. $y = f(x) + c$ B. $y = f(x) - c$ C. $y = f(x - c)$ D. $y = f(x + c)$

Final answer:

C

(see page 129 of the book)

Problem 6

The derivative of the function $f(x) = (5x + 2)\left(\frac{3}{x^2} + \frac{1}{x}\right)$ is

- A. $-17x^{-2} - 12x^{-3}$ B. $-12x^{-2} - 17x^{-3}$
C. $-17x^{-2} + 12x^{-3}$ D. $-15x^{-2} - 13x^{-3}$

Final answer:

A

We first simplify $f(x)$ before we compute the derivative:

$$f(x) = 15x^{-1} + 5 + 6x^{-2} + 2x^{-1} = 17x^{-1} + 5 + 6x^{-2}$$

$$f'(x) = -17x^{-2} - 12x^{-3}$$

Problem 7

Calculate the derivative of $f(x) = (5\ln x + x^2)^2$. The answer is:

A. $2(5\ln x + 2x^2)\left(\frac{5}{x} + 2x\right)$

B. $2(5\ln x + x^2)\left(\frac{5}{x} + 2\right)$

C. $2(5\ln x + x^2)\left(\frac{5}{x} + 2x\right)$

D. $2(5\ln x + 2x)\left(\frac{5}{x} + 2\right)$

Final answer:

C

$$f'(x) = 2(5\ln x + x^2)\left(\frac{5}{x} + 2x\right)$$

Problem 8

Let $f(x) = 7x^2 + 5x - 8$. The interval on which this function is increasing is

A. $\left[0, -\frac{5}{14}\right]$

B. $\left(-\infty, -\frac{5}{14}\right]$

C. $\left[-\frac{5}{14}, +\infty\right)$

D. $\left[-\frac{5}{7}, +\infty\right)$

Final answer:

C

$$f'(x) = 14x + 5$$

The function is increasing when $f'(x) \geq 0$, so that

$$14x + 5 \geq 0 \Leftrightarrow 14x \geq -5 \Leftrightarrow x \geq -\frac{5}{14} \Leftrightarrow x \in \left[-\frac{5}{14}, +\infty\right)$$

Problem 9

Determine the first derivative of y at the point $(x,y)=(1,-2)$ of $x^3 - xy + y^5 = -29$ by implicit differentiation. Which of the following answers is correct?

A. $-\frac{5}{19}$

B. $-\frac{5}{39}$

C. $\frac{5}{59}$

D. $-\frac{5}{79}$

Final answer:

D

$$y' = -\frac{F'_x}{F'_y} = -\frac{3x^2 - y}{-x + 5y^4}, \text{ by substituting in } (x,y)=(1,-2), \text{ we have}$$

$$y' = -\frac{3(1)^2 + 2}{-1 + 5(-2)^4} = -\frac{5}{79}$$

Problem 10

Consider the function $F(x, y, z) = \left(2\sqrt{x} + \frac{5x}{\sqrt{y}} + \frac{y}{\sqrt{x}}\right)^6$. The degree of homogeneity is

A. $\frac{1}{2}$

B. 2

C. 3

D. 4

Final answer:

C

$$F(tx, ty, tz) = \left(2\sqrt{tx} + \frac{5tx}{\sqrt{ty}} + \frac{ty}{\sqrt{tx}}\right)^6 = \left(\sqrt{t}\left(2\sqrt{x} + \frac{5x}{\sqrt{y}} + \frac{y}{\sqrt{x}}\right)\right)^6 = t^3\left(2\sqrt{x} + \frac{5x}{\sqrt{y}} + \frac{y}{\sqrt{x}}\right)^6 = t^3 F(x, y, z)$$

Therefore, the degree of homogeneity is 3.

Part III: Calculation problems

5 points per problem

Problem 11

Determine the remainder of the polynomial division $\frac{x^3 - 18x^2 + 11}{x - 3}$.

Final answer:

-124

$$(x^3 - 18x^2 + 0x + 11) \div (x - 3) = x^2 - 15x - 45$$

$$\begin{array}{r} x^3 - 3x^2 \\ \hline -15x^2 + 0x \\ -15x^2 + 45x \\ \hline -45x + 11 \\ -45x + 135 \\ \hline -124 \end{array} \quad \begin{array}{l} \leftarrow x^2(x-3) \\ \\ \leftarrow -15x(x-3) \\ \\ \leftarrow -45(x-3) \\ \\ \text{remainder} \end{array}$$

Therefore, the remainder of the polynomial division is -124.

Problem 12

Consider the function

$$f(x) = \begin{cases} \frac{x^2 + x - 12}{x + 4} & \text{if } x \neq -4 \\ c & \text{if } x = -4 \end{cases}$$

What should be the value of c such that f is continuous at -4 ?

Final answer:

$$c = -7$$

The function is continuous at -4 when $\lim_{x \rightarrow -4} f(x) = f(-4) = c$.

$$\text{As } \lim_{x \rightarrow -4} f(x) = \lim_{x \rightarrow -4} \frac{(x+4)(x-3)}{(x+4)} = -7, c \text{ should be } -7$$

Problem 13

Find the equation for the tangent line to the graph of $f(x) = e^x + 5$ at the point $(0, 6)$

Final answer:

$$y = x + 6$$

First, by differentiating w.r.t. x , we get

$$f'(x) = e^x \text{ so that the slope of the tangent line at point } (0,6) \text{ is } f'(0) = e^0 = 1$$

Therefore, the equation of the tangent line is $y - 6 = 1(x - 0) \Leftrightarrow y = x + 6$

Problem 14

Consider the demand function $D(p)$ of a good that gives demand as function of price p . Suppose that when $p=100$, demand $D(100)$ is equal to 400 and the price elasticity of demand is equal to -15. What will (approximately) be the level of demand when price p increases to 101?

Final answer:

340

$El_x D(p=100) = -15$, so when price increases by 1%, i.e. from 100 to 101, demand will decrease by approximately 15%. Because $400 \times 15\% = 60$, it follows that $D(101) \approx 400 - 60 = 340$.

Problem 15

Find the maximum and minimum value of the function $f(x) = 2x^2 + \frac{32}{x^2}$ on the interval $[-3, 3]$.

Final answer:

The correct answer is that the function does not have a maximum value (because $\lim_{x \rightarrow 0} f(x) = \infty$) and the minimum value is 16 at $x = \pm 2$ (see calculations below).

However, because the question is ill-posed (as the function is not defined for $x=0$), all answers will be considered correct.

1) Stationary points

$$f' = 4x + \left(-\frac{64}{x^3}\right) = \frac{4x^4 - 64}{x^3} = \frac{4(x^4 - 16)}{x^3} = 0 \Rightarrow x^4 = 16 \Rightarrow x = \pm 2$$

2) Endpoints $x = -3, x = 3$

$$3) f(-3) = f(3) = 21\frac{5}{9}, f(-2) = f(2) = 16$$

So the minimum value is 16 at $x = \pm 2$.

Problem 16

Let $f(x, y) = 2x^2 + y^x + e^{3x}$. Find all first- and second-order partial derivatives.

Final answer:

$$f'_x = 4x + y^x \ln y + 3e^{3x}$$

$$f''_{xx} = 4 + y^x (\ln y)^2 + 9e^{3x}$$

$$f'_y = xy^{x-1}$$

$$f''_{yy} = x(x-1)y^{x-2}$$

$$f''_{xy} = f''_{yx} = y^{x-1} + xy^{x-1} \ln y$$

Part IV: Open problems

8 points per problem

Problem 17 (2+2+2+2=8 points)

Consider the function $f(x) = 2x^3 - 8x^2 + 8x + 3$.

a) Determine the first derivative and the stationary point(s) of f .

$$f' = 6x^2 - 16x + 8 = 2(3x^2 - 8x + 4) = 2(3x - 2)(x - 2)$$

$$f' = 0 \Rightarrow x = 2 \text{ or } \frac{2}{3}$$

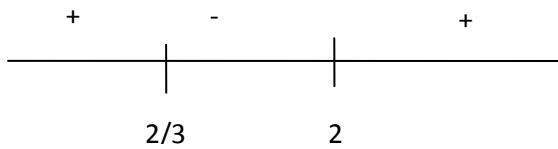
b) Classify the stationary point(s) using the first derivative (local minimum, local maximum or not an extreme point).

$$f'(0) = 8 > 0$$

$$f'(1) = 6 - 16 + 8 < 0$$

$$f'(3) = 54 - 48 + 8 > 0$$

So the sign diagram of f' is:



Therefore $x = \frac{2}{3}$ is a local maximum point, $x = 2$ is a local minimum point.

c) Determine the second derivative f'' and all value(s) of x for which $f''(x) = 0$.

$$f'' = 12x - 16 = 0 \Rightarrow x = \frac{4}{3}$$

d) Determine on which interval(s) the function f is concave.

We pick the following 2 points to determine the signs of f''

$$f''(0) = -16 < 0$$

$$f''(2) = 8 > 0$$

f is concave when $f'' \leq 0$, therefore, f is concave on the interval $(-\infty, \frac{4}{3}]$

Problem 18 (2+2+1+3=8 points)

Consider the function $f(x, y) = 2x^3y - 54y - 3x^2 + 10$.

a) Calculate all first order partial derivatives.

$$f'_x = 6x^2y - 6x$$

$$f'_y = 2x^3 - 54$$

b) Calculate all second order partial derivatives.

$$f''_{xx} = 12xy - 6$$

$$f''_{yy} = 0$$

$$f''_{xy} = f''_{yx} = 6x^2$$

c) Find all stationary points of f .

$$f'_x = 0 \Rightarrow x = 0 \text{ or } xy = 1$$

First assume $x = 0$, then $f'_y = 0 \Rightarrow 0 - 54 = 0$ which is impossible.

Now assume $x \neq 0$, then $f'_x = 0$ implies $y \neq 0$ so we can take $x = \frac{1}{y}$. Substitution in

$$f'_y = 0 \Rightarrow \left(\frac{1}{y}\right)^3 = 27 \Rightarrow y = \frac{1}{3}, x = 3$$

So there is only one stationary point, which is $(3, \frac{1}{3})$.

d) Calculate the value of f in every stationary point and use the second-derivative test to classify them (local minimum, local maximum, saddle point or inconclusive).

$$f\left(3, \frac{1}{3}\right) = -17$$

Let $A = f''_{xx}$, $B = f''_{xy}$, $C = f''_{yy}$ where

$$f''_{xx} = 12xy - 6$$

$$f''_{yy} = 0$$

$$f''_{xy} = 6x^2$$

| Stationary points | A | B | C | $AC - B^2$ | Classification |
|--------------------|-----|-----|-----|------------|----------------|
| $(3, \frac{1}{3})$ | 6 | 54 | 0 | <0 | Saddle point |

Problem 19 (2+3+3=8 points)

We want to find the local extreme values of $f(x, y) = 6xy^2 - 18y$ subject to $g(x, y) = x^2 - y^2 = 0$.

- a) Give the Lagrangian function and the first order conditions of the Lagrange multiplier method for this particular problem.

The Lagrangian function is given by

$$L = 6xy^2 - 18y - \lambda(x^2 - y^2)$$

The first order conditions are:

$$L'_x = 6y^2 - 2\lambda x = 0 \quad (1)$$

$$L'_y = 12xy - 18 + 2\lambda y = 0 \quad (2)$$

$$x^2 - y^2 = 0 \quad (3)$$

- b) Find the point(s) (x, y) and the corresponding value(s) of λ that satisfy the first order conditions of the Lagrange multiplier method.

From equation (3), we get

$$(x + y)(x - y) = 0 \Rightarrow x = \pm y$$

- In case $x = y$, substitute it into (1), we get

$$6y^2 = 2\lambda y \Rightarrow y = 0 \quad \text{or} \quad y = \frac{\lambda}{3}$$

Substitute $y = 0$ into (2), we get $-18=0$, which is impossible.

Substitute $y = \frac{\lambda}{3}$ into (2), we get the following solution

$$(x, y, \lambda) = (1, 1, 3) \quad \text{or} \quad (-1, -1, -3)$$

- In case $x = -y$, substitute it into (1), we get

$$y(6y + 2\lambda) = 0 \Rightarrow y = 0 \quad \text{or} \quad y = -\frac{\lambda}{3}$$

Substitute $y = 0$ into (2), we get $-18=0$, which is impossible.

Substitute $y = -\frac{\lambda}{3}$ into (2), we get $y^2 = -1$, indicating there is no solution.

Together we find 2 points that satisfy the first-order conditions:

$$(x, y, \lambda) = (1, 1, 3) \quad \text{or} \quad (-1, -1, -3)$$

c) What are the local extreme values and at which point(s) are they attained?

We have

$$f''_{11}(x, y) = 0, f''_{12}(x, y) = 12y, f''_{22}(x, y) = 12x,$$

$$g'_1(x, y) = 2x, g'_2(x, y) = -2y,$$

$$g''_{11}(x, y) = 2, g''_{12}(x, y) = 0, g''_{22}(x, y) = -2$$

Substituting these functions in

$$D(x, y, \lambda) = (f''_{11} - \lambda g''_{11})(g'_2)^2 - 2(f''_{12} - \lambda g''_{12}) \cdot g'_1 \cdot g'_2 + (f''_{22} - \lambda g''_{22})(g'_1)^2$$

and evaluating this expression in (1,1) and (-1,-1) gives

$$D(1, 1, 3) = 144 > 0 \quad \text{and} \quad D(-1, -1, -3) = -144 < 0$$

Hence, we find that (-1,-1) corresponds to a local maximum, while (1,1) corresponds to a local minimum, with $f(1,1) = -12$ and $f(-1,-1) = 12$.