This exam consisted of 5 numbered pages.

ERASMUS UNIVERSITY ROTTERDAM
Erasmus School of Economics
International Bachelor Economics & Business Economics, Bachelor 1

# Exam Mathematics 1 (FEB11003X-10)

Friday 29 October 2010, 9:30 – 12:30 hrs

## Instructions ESE

- You are not allowed to use a calculator.
- You are not allowed to use a programmable calculator.
- You are not allowed to use notes (except a cheat sheet, see below).
- You are not allowed to use books.
- You are not allowed to use a dictionary.
- You are allowed to take the examination papers with you (but you have to turn in the separate answer sheet).

## Additional information

You are allowed to use a cheat sheet, for which the following rules apply:

- Two-sided A4
- Your name and student number in the upper right corner

All material that does not satisfy these rules will be taken away from you and may be considered a fraud attempt.

## Use the separate answer sheet to indicate your answers.

The exam consists of 19 problems grouped in four parts, each with different types of problems. For the basic problems, multiple choice problems and calculation problems you will score 3, 4 and 5 points, respectively, per correct answer and no points for incorrect answers. For the open problems (8 points per problem) your score will depend on the answer and the calculation. The exam grade is the result from the formula (10 + number of scored points)/10. So 63 scored points result in grade 7.3, since (10+63)/10= 7.3.

## Good luck!

# Part I: Basic problems

3 points per problem

## Problem 1

Suppose $f(x)$ is an even function (i.e., a function of which the graph is symmetric about the $y$-axis). Now consider the function $f(x+d)$, where $d$ is some positive constant. Is it possible that this function is also even? Explain your answer.

*Final answer:*

Yes, when $f(x) = 0$ (or another constant)

When $f(x) = 0$, $f(-x) = 0 = f(x)$, so it is an even function.

As $f(x+d) = 0$, $f(-x+d) = 0 = f(x+d)$, it is still an even function.

You could also prove this by shifting the graph of $f(x) = 0$ (or another constant) to the left by $d$ units. It can be seen that the graph of $f(x+d)$ is symmetric about the $y$-axis as well.

## Problem 2

Suppose that the function $f$ with domain $D$ has an inverse function $f^{-1}$. Does this imply that $f$ is either strictly increasing or strictly decreasing on $D$? Explain your answer.

*Final answer:*

In general, the answer is "No". For example, consider the function

$f(x) = \begin{cases} x-1 & x \in [0,+1) \\ x+1 & x \in [-1,0) \end{cases}$, the inverse function is $f^{-1}(x) = \begin{cases} x-1 & x \in [0,+1) \\ x+1 & x \in [-1,0) \end{cases}$. As $f(-0.5) = 0.5,\ f(0) = -1,\ f(0.5) = -0.5$, the function is neither strictly increasing nor decreasing in its domain.

(Note that the above function is discontinuous. If we would consider only continuous functions, the answer would be "Yes". As the original function has an inverse function, it must be one-to-one. In other words, there doesn't exist two different $x$ in the domain, which corresponds to the same function value $y$. If a continuous function is neither strictly increasing nor decreasing, there must exist two different points in its domain with the same corresponding function value, then one-to-one feature of the function is violated. )

## Problem 3

Let $f$ be a function that is differentiable on its domain. Give an example (i.e., you have to specify a specific function) that shows that a stationary point of $f$ is not necessarily an extreme point.

*Final answer:*

For example $f(x) = x^3$

$f'(x) = 3x^2$, so that the stationary point is 0.

By drawing a sign diagram of first derivative around 0, we find the derivative is positive on both sides of the stationary point, so that it is not an extreme point.


Any other example that could show *f′(x)* doesn't change sign about the stationary point, thus not an extreme point, is acceptable.

## Problem 4

We know that a continuous function with a domain that is a closed and bounded interval always has a minimum and a maximum. Show that a function that is not continuous does not necessarily have a minimum and a maximum on a closed and bounded interval.

*Final answer:*

A counter example is $f(x) = \begin{cases} \frac{1}{x}, x \in [-1,0) \\ 0, x = 0 \end{cases}$

This function is defined on a closed and bounded domain, but there is no minimum within its domain.

Any other example that could show there is no maximum or minimum on a closed bounded interval is acceptable.

## Part II: Multiple choice problems

4 points per problem

### Problem 5

Solve $\frac{1}{9}3^{2x^2+4x} = \left(3^x\right)^{2x}$ for $x$. Which of the following answers is correct?

A. $x = 0$ or $x = 2$ B. $x = \frac{1}{2}$ C. $x = 2$ D. $x = 0$ or $x = \frac{1}{2}$

*Final answer:*

B

$$\frac{1}{9}3^{2x^2+4x} = \left(3^x\right)^{2x} \Leftrightarrow 3^{-2+2x^2+4x} = 3^{2x^2} \Leftrightarrow 2x^2 + 4x - 2 = 2x^2 \Leftrightarrow x = \frac{1}{2}$$

### Problem 6

The derivative of the function $f(x) = (3x+1)\left(\frac{1}{x^2} + \frac{1}{x}\right)$ is

A $-\frac{4x+2}{x^3}$ B. $-3\left(\frac{2}{x^3} + \frac{1}{x^2}\right)$

C. $\frac{3}{x^2} + \frac{3}{x}$ D. $-\frac{2}{x^3} - \frac{7}{x^2} - \frac{3}{x}$

*Final answer:*

A

We first simplify *f(x)* before we compute the derivative:

$$f(x) = 3x^{-1} + 3 + x^{-2} + x^{-1} = 4x^{-1} + 3 + x^{-2}$$

$$f'(x) = -4x^{-2} - 2x^{-3} = -\frac{4x+2}{x^3}$$

### Problem 7

Calculate the derivative of $f(x) = (\ln x + x)^2$. The answer is:

A. $2\left(\frac{\ln x}{x} + x\right)$ B. $2\left(\frac{1}{x} + x\right)$ C. $2(\ln x + x)\left(\frac{1}{x} + 1\right)$ D. $2\left(\frac{1}{x} + 1\right)$

*Final answer:*

| C |
|---|
| $f'(x) = 2(\ln x + x)(\frac{1}{x} + 1)$ |

## Problem 8

Let $f(x) = \frac{2x}{x^2 + 2}$. The interval on which this function is increasing is

A. $[0, \sqrt{2}]$ B. $(-\infty, \sqrt{2}]$ C. $[-\sqrt{2}, \sqrt{2}]$ D. $[\sqrt{2}, \infty)$

| *Final answer:*<br><br>C |
|---|
| $f'(x) = \frac{2(x^2 + 2) - 2x(2x)}{(x^2 + 2)^2} = \frac{-2x^2 + 4}{(x^2 + 2)^2}$<br><br>The function is increasing when $f'(x) \geq 0$, so that<br><br>$\frac{-2x^2 + 4}{(x^2 + 2)^2} \geq 0 \Leftrightarrow -2x^2 + 4 \geq 0 \Leftrightarrow x^2 \leq 2 \Leftrightarrow x \in [-\sqrt{2}, \sqrt{2}]$ |

## Problem 9

Let $f(x, y) = \sqrt{\ln(x^2) - \ln(y^2)}$ where $x > 0$ and $y > 0$. Calculate the partial elasticity of $f$ with respect to $y$.

Which of the following answers is correct?

A. $\frac{1}{\ln(x^2) - \ln(y^2)}$ B. $-\frac{2}{\ln(x^2) - \ln(y^2)}$

C. $\frac{1}{\ln x - \ln y}$ D. $-\frac{1}{2\ln x - 2\ln y}$

| *Final answer:*<br><br>D |
|---|

$$f_y'(x,y)=\frac{1}{2}(2\ln x-2\ln y)^{-\frac{1}{2}}(-\frac{2}{y})=-\frac{1}{y}(2\ln x-2\ln y)^{-\frac{1}{2}}$$

The partial elasticity is

$$El_y f=\frac{y}{f(x,y)}f_y'(x,y)=\frac{y}{\sqrt{2\ln x-2\ln y}}[-\frac{1}{y}(2\ln x-2\ln y)^{-\frac{1}{2}}]=-\frac{1}{2\ln x-2\ln y}$$

### Problem 10

Consider the function $F(x,y,z,)=\left(\sqrt{x}+\frac{2x}{\sqrt{y}}+7\frac{y}{\sqrt{x}}\right)^4$. The degree of homogeneity is

A. $\frac{1}{2}$ B. 1 C. 2 D. 4

*Final answer:*

C

$$F(tx,ty,tz,)=\left(\sqrt{tx}+\frac{2tx}{\sqrt{ty}}+7\frac{ty}{\sqrt{tx}}\right)^4=\left(\sqrt{t}(\sqrt{x}+\frac{2x}{\sqrt{y}}+7\frac{y}{\sqrt{x}})\right)^4=t^2(\sqrt{x}+\frac{2x}{\sqrt{y}}+7\frac{y}{\sqrt{x}})^4$$

$$=t^2F(x,y,z)$$

Therefore, the degree of homogeneity is 2.

# Part III: Calculation problems

5 points per problem

### Problem 11

Determine the remainder of the polynomial division $\frac{x^3-12x^2-42}{x-3}$.

*Final answer:*

-123

$$(x^3 - 12x^2 + 0x - 42) \div (x-3) = x^2 - 9x - 27$$

$$\begin{array}{lll}
\underline{x^3 - 3x^2} & & \leftarrow x^2(x-3) \\
\quad -9x^2 + 0x & & \\
\quad \underline{-9x^2 + 27x} & & \leftarrow -9x(x-3) \\
\qquad -27x - 42 & & \\
\qquad \underline{-27x + 81} & & \leftarrow -27(x-3) \\
\qquad\quad -123 & & \textit{remainder}
\end{array}$$

Therefore, the remainder of the polynomial division is -123.

## Problem 12

Consider the function $y = 3\ln(x-3) + 7$ with domain [4, ∞). Determine the inverse and its domain. If the inverse does not exist, report 'no inverse'.

*Final answer:*

The inverse function is $x = e^{\frac{y-7}{3}} + 3$, and the domain is $[7,+\infty)$

$y = 3\ln(x-3) + 7 \Leftrightarrow$

$y - 7 = 3\ln(x-3) \Leftrightarrow$

$\ln(x-3) = \dfrac{y-7}{3} \Leftrightarrow$

$x - 3 = e^{\frac{y-7}{3}} \Leftrightarrow$

$x = e^{\frac{y-7}{3}} + 3$

The domain of the inverse is the range of the origin.

As $y = 3\ln(x-3) + 7$ is strictly increasing and $x \in [4,+\infty)$,

$x \geq 4 \Rightarrow x - 3 \geq 1 \Rightarrow \ln(x-3) \geq \ln 1 = 0 \Rightarrow 3\ln(x-3) + 7 \geq 3 \times 0 + 7 = 7$, therefore, the domain of the inverse is

$y \in [7,+\infty)$

## Problem 13

Consider the equation $5xy + 3y^2 - x^2 y = 7$. Use implicit differentiation to compute the value of $y'$ in the point $(x, y) = (1,1)$.

| *Final answer:* $$-\frac{3}{10}$$ |
|---|
| First, by differentiating both sides of equation w.r.t. x, we get<br>$5y+5xy'+6yy'-2xy-x^2y'=0 \Leftrightarrow$<br>$y'(5x+6y-x^2)=2xy-5y \Leftrightarrow$<br>$y'=\frac{2xy-5y}{5x+6y-x^2}$<br>Therefore, in the point $(x,y)=(1,1).$, $y'=\frac{2xy-5y}{5x+6y-x^2}=\frac{2-5}{5+6-1}=-\frac{3}{10}$ |

**Problem 14**

Consider a demand function *D*(*p*) that gives demand as function of price *p*. Suppose that when *p*=100, demand is equal to 350 and the price elasticity of demand is equal to -12. What will (approximately) be the level of demand when price *p*=101?

| *Final answer:* 308 |
|---|
| $El_x D(p=100)=-12$, so when price increases by 1%, i.e. from 100 to 101, demand will decrease by 12%. In this case we can compute the decrease in demand, which equals to $350\times 12\%=42$ . Hence $D(101)=350-42=308$. |

**Problem 15**

Find the maximum and minimum value of the function $f(x)=x^2+\frac{16}{x}$ on the interval [1, 3].

| *Final answer:* |
|---|
| The maximum value is 17 at $x=1$, the minimum value is 12 at $x=2$. |

1) Stationary points

$$f' = 2x + \left(-\frac{16}{x^2}\right) = \frac{2x^3 - 16}{x^2} = \frac{2(x^3 - 8)}{x^2} = 0 \Rightarrow x^3 = 8 \Rightarrow x = 2$$

2) Endpoints $x = 1,\ x = 3$

3) $f(1) = 17,\ f(2) = 12,\ f(3) = 14\frac{1}{3}$

So the maximum value is 17 at $x = 1$, the minimum value is 12 at $x = 2$.

**Problem 16**

Let $f(x, y) = 2xy + y^x + \ln(3x^2 + 1)$. Find all first- and second-order partial derivatives.

*Final answer:*

$$f'_x = 2y + y^x \ln y + \frac{6x}{3x^2 + 1}$$

$$f''_{xx} = y^x (\ln y)^2 + \frac{6(3x^2 + 1) - 6x(6x)}{(3x^2 + 1)^2} = y^x (\ln y)^2 + \frac{6 - 18x^2}{(3x^2 + 1)^2}$$

$$f'_y = 2x + xy^{x-1}$$

$$f''_{yy} = x(x-1)y^{x-2}$$

$$f''_{xy} = f''_{yx} = 2 + xy^{x-1} \ln y + y^x \frac{1}{y} = 2 + xy^{x-1} \ln y + y^{x-1}$$

# Part IV: Open problems

8 points per problem

**Problem 17 (2+2+2+2=8 points)**

Consider the function $f(x) = x^5 - 5x^4 + 5x^3 + 3$.

a) Determine the first derivative and the stationary point(s) of $f$.

$f' = 5x^4 - 20x^3 + 15x^2 = 5x^2(x^2 - 4x + 3) = 5x^2(x-1)(x-3)$

$f' = 0 \Rightarrow x = 0, 1, or\, 3$

b) Classify the stationary point(s) using the first derivative (local minimum, local maximum or not an extreme point).

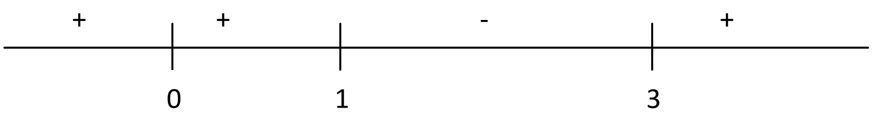
$f'(-1) = 5(-2)(-4) > 0$

$f'(\frac{1}{2}) = \frac{5}{4}\left(-\frac{1}{2}\right)\left(-\frac{5}{2}\right) > 0$

$f'(2) = 20 \cdot 1 \cdot (-1) < 0$

$f'(4) = 5 \cdot 16 \cdot 3 \cdot 1 > 0$

So the sign diagram of $f'$ is:




Therefore $x = 0$ is not an extreme point, $x = 1$ is a local maximum point, $x = 3$ is a local minimum point.

c) Determine the second derivative $f''$ and all value(s) of $x$ for which $f''(x) = 0$.

$f'' = 20x^3 - 60x^2 + 30x = 10x(2x^2 - 6x + 3)$

$f'' = 0 \Rightarrow x = 0 \;or\; 2x^2 - 6x + 3 = 0 \Rightarrow x = 0 \;or\; \frac{3 \pm \sqrt{3}}{2}$

d) Determine on which interval(s) the function $f$ is concave. (It may be useful to know that $\sqrt{2} \approx 1.4$, $\sqrt{3} \approx 1.7$, $\sqrt{5} \approx 2.2$ and $\sqrt{7} \approx 2.6$)

We use $\sqrt{3} \approx 1.7$ to get $\frac{3-\sqrt{3}}{2} \approx 0.65$ and $\frac{3+\sqrt{3}}{2} \approx 2.35$, so we could pick the following 4 points to determine the signs of $f''$

$$f''(-1) = -10(2+6+3) < 0$$
$$f''(0.5) = 5(0.5-3+3) > 0$$
$$f''(1) = 10(2-6+3) < 0$$
$$f''(3) = 30(18-18+3) > 0$$

So $f'' \leq 0$ , i.e. $f$ is concave on the intervals $(-\infty, 0]$ and $[\frac{3-\sqrt{3}}{2}, \frac{3+\sqrt{3}}{2}]$.

## Problem 18 **(2+2+1+3=8 points)**

Consider the function $f(x, y) = x^2y^2 - 4y^2 - 9x^2 + 49$.

a) Calculate all first order partial derivatives.

$$f'_x = 2xy^2 - 18x = 2x(y^2 - 9)$$
$$f'_y = x^2 2y - 8y = 2y(x^2 - 4)$$

b) Calculate all second order partial derivatives.

$$f''_{xx} = 2(y^2 - 9)$$
$$f''_{yy} = 2(x^2 - 4)$$
$$f''_{xy} = f''_{yx} = 2x2y = 4xy$$

c) Find all stationary points of *f*.

$f'_x = 0 \Rightarrow x = 0 \text{ or } y = \pm 3$

Substitute $x = 0$ into $f'_y = 0 \Rightarrow 2y(0-4) = 0 \Rightarrow y = 0$

Substitute $y = 3$ into $f'_y = 0 \Rightarrow 6(x^2 - 4) = 0 \Rightarrow x = \pm 2$

Substitute $y = -3$ into $f'_y = 0 \Rightarrow -6(x^2 - 4) = 0 \Rightarrow x = \pm 2$

So there are five stationary points (0, 0), (2, 3), (-2, 3), (2, -3), (-2, -3).

d) Calculate the value of *f* in every stationary point and use the second-derivative test to classify them (local minimum, local maximum, saddle point or inconclusive).

$f(0,0) = 49$

$f(2,3) = f(2,-3) = f(-2,3) = f(-2,-3) = 36 - 36 - 36 + 49 = 13$

Let $A = f''_{xx}, B = f''_{xy}, C = f''_{yy}$ where

$f''_{xx} = 2(y^2 - 9)$

$f''_{yy} = 2(x^2 - 4)$

$f''_{xy} = 4xy$

| Stationary points | $A$ | $B$ | $C$ | $AC - B^2$ | Classification |
|---|---|---|---|---|---|
| (0, 0) | -18 < 0 | 0 | -8 | >0 | Local maximum point |
| (2, 3) | 0 | 24 | 0 | <0 | Saddle point |
| (2, -3) | 0 | -24 | 0 | <0 | Saddle point |
| (-2, 3) | 0 | -24 | 0 | <0 | Saddle point |
| (-2, -3) | 0 | 24 | 0 | <0 | Saddle point |

## Problem 19 (2+3+3=8 points)

We want to find the maximum and minimum values of $f(x,y) = 5x^2y$ subject to $g(x,y) = x^2 + y^2 = 9$. Note that the minimum and maximum values exist; because f is a continuous function and the constraint curve is closed and bounded (it is a circle).

a) Give the Lagrangian function and the first order conditions of the Lagrange multiplier method for this particular problem.

The Lagrangian function is given by

$$L = 5x^2 y - \lambda(x^2 + y^2 - 9)$$

The first order conditions are:

$$L'_x = 10xy - 2\lambda x = 0 \quad (1)$$
$$L'_y = 5x^2 - 2\lambda y = 0 \quad (2)$$
$$x^2 + y^2 = 9 \quad (3)$$

b) Find the point(s) $(x,y)$ and the corresponding value(s) of $\lambda$ that satisfy the first order conditions of the Lagrange multiplier method.

From equation (1), we get

$$2x(5y - \lambda) = 0 \Rightarrow x = 0 \text{ or } \lambda = 5y$$

- In case $x = 0$, substitute it into (3), we get $y^2 = 9 \Rightarrow y = \pm 3$

Substitute the $x$, $y$ values into (2), we get $\lambda = 0$.

- In case $\lambda = 5y$, substitute it into (2), we get

$$5x^2 - 10y^2 = 0 \Rightarrow x^2 = 2y^2 \quad (4)$$

Substitute (4) into (3), we get $3y^2 = 9 \Rightarrow y = \pm\sqrt{3}$

Plug $y$ values back into (4), we get $x^2 = 6 \Rightarrow x = \pm\sqrt{6}$

Plug $y$ values back into $\lambda = 5y$, we can get the according $\lambda$ values.

Together we find 6 points that satisfying the first-order conditions:

$(0, \pm 3)$ with $\lambda = 0$, $(\pm\sqrt{6}, \sqrt{3})$ with $\lambda = 5\sqrt{3}$, and $(\pm\sqrt{6}, -\sqrt{3})$ with $\lambda = -5\sqrt{3}$.

c) What are the maximum and minimum values and at which point(s) are they attained?

Compute the function values at the six candidate points:

$$f(\sqrt{6}, \sqrt{3}) = f(-\sqrt{6}, \sqrt{3}) = 30\sqrt{3}$$
$$f(\sqrt{6}, -\sqrt{3}) = f(-\sqrt{6}, -\sqrt{3}) = -30\sqrt{3}$$
$$f(0, 3) = f(0, -3) = 0$$

So the maximum value of *f* is $30\sqrt{3}$ at $(\sqrt{6}, \sqrt{3})$ *and* $(-\sqrt{6}, \sqrt{3})$;

The minimum value of *f* is $-30\sqrt{3}$ at $(\sqrt{6}, -\sqrt{3})$ *and* $(-\sqrt{6}, -\sqrt{3})$.