Mathematics 1 FEB11003X
Model exam WITH ANSWERS

This model exam consists of problems that have been part of actual exams in the past. It has the same structure as the exam that you will take. There are four parts, each with different types of problems. A maximum of 90 points can be scored. The grade for this written exam is the result from the formula (10 + number of scored points)/10. All problems should (and can) be made without the use of an electronic calculator.

Part I Basic Problems

3 points per problem

Problem 1

Explain whether or not it is necessary that a function $f(x)$ is defined at x=a when we want to compute $\lim_{x \to a} f(x)$.

ANSWER:

No this is not necessary, because when taking the limit we look at the functions values for x very close to a, but not at $x=a$ itself. (See also the example on page 177 of the book.)

Problem 2

Describe in words the meaning of the elasticity of a function $f(x)$ with respect to x.

ANSWER:

The elasticity gives approximatley the percentual change of $f(x)$ when x is increased by 1%.

Problem 3

Consider the function $f(x,y) = (x+y)^2$. What can you say about the sign (≤0 or ≥0) of f''_{11}, f''_{22} and $f''_{11} \cdot f''_{22} - (f''_{12})^2$? Is this function concave, convex, or neither?

ANSWER:

$f''_{11} = 2$, $f''_{22} = f''_{12} = 2$, so $f''_{11} \geq 0$, $f''_{22} \geq 0$ and $f''_{11} \cdot f''_{22} - (f''_{12})^2 \geq 0$, which means that the function is convex.

Problem 4

Consider a function $f(x,y)$ where both x and y are non-negative. Explain whether or not such a function has always a maximum and a minimum on its domain.

ANSWER:

Such a function does not always have a maximum and a minimum. Consider, for instance, $f(x, y) = x - y$ for $x \geq 0, y \geq 0$, then there is no largest function value (one can take x always larger while y=0) and no smallest vallue (one can take y in absolute value larger while x=0).

Part II Multiple Choice Problems

4 points per problem

Problem 5

Solve $2^{3t}4^t = 16$ for t. Which of the following answers is correct?

A. $\frac{1}{2}$ B. $\frac{4}{5}$ C. 1 D. $\frac{2}{3}$

CORRECT ANSWER: B

$$2^{3t}4^t = 16 \Leftrightarrow 2^{3t}2^{2t} = 2^4 \Leftrightarrow 2^{3t+2t} = 2^4 \Leftrightarrow 2^{5t} = 2^4 \Leftrightarrow 5t = 4 \Leftrightarrow t = \frac{4}{5}$$

Problem 6

Given a function $f(x)$ and its inverse function $g(x)$. Which of the following statements is correct?

A. $f(g(x)) = 1$ B. $g(f(x)) = 1$ C. $f(g(x)) = x$ D. $f(g(x)) = \frac{1}{x}$

CORRECT ANSWER: C

Because $g(x)$ is the inverse of $f(x)$, for every $y = f(x)$ it holds $x = g(y)$. Therefore, $g(f(x)) = g(y) = x$.

Problem 7

Find the elasticity $El_x y$ of $y = x^3 e^{2x+1}$. Which of the following answers is correct?

A. $3+2x$ B. $3+x$ C. $\frac{3}{x}+2$ D. $2+3x$

CORRECT ANSWER: A

$$\frac{x}{x^3 e^{2x+1}} \cdot \left(3x^2 \cdot e^{2x+1} + x^3 \cdot 2e^{2x+1}\right) = \frac{x^3 e^{2x+1}}{x^3 e^{2x+1}}(3+2x) = 3+2x$$

Problem 8

Consider the function $f(x, y) = x^3 - x^2 - 3x + 2xy^2 - y^4$.

The number of stationary points is:

A. 0 B. 1 C. 2 D. 3

CORRECT ANSWER: D

$f'_x = 3x^2 - 2x - 3 + 2y^2 \quad f'_y = 4xy - 4y^3$

$f'_y = 0 \Rightarrow 4y(x - y^2) = 0 \Rightarrow y = 0$ or $y^2 = x$

1) Substitution of $y = 0$ in $f'_x = 0$ leads $3x^2 - 2x - 3 = 0 \overset{\text{quadratic formula}}{\Longrightarrow} x_{1,2} = \frac{2 \pm \sqrt{4+36}}{6}$

2) Substitution of $y^2 = x$ in $f'_x = 0$ leads $3x^2 - 2x - 3 + 2x = 0 \Leftrightarrow 3x^2 - 3 = 0 \Leftrightarrow x_{1,2} = \pm 1$

From 1) results 2 stationary points, from 2) results 1 stationary point, because $y^2 = x$ has to be greater than or equal to zero.

Hence, the total number of stationary points is 3.

Problem 9

$f(x, y) = \sqrt{\ln x^2 + \ln y}$

Calculate the partial elasticity of f with respect to x and the partial elasticity of f with respect to y. Divide the first by the second.

Which of the following answers is correct?

A. $\frac{2y}{x}$ B. 2 C. $\frac{1}{x}$ D. $\frac{y}{x^2}$

CORRECT ANSWER: B

Note that $\ln x^2 = 2 \ln x$.

$El_x = \frac{x}{\sqrt{2\ln x + \ln y}} \cdot \frac{\frac{2}{x}}{2\sqrt{2\ln x + \ln y}} = \frac{2}{2(2\ln x + \ln y)}$

$El_y = \frac{y}{\sqrt{2\ln x + \ln y}} \cdot \frac{\frac{1}{y}}{2\sqrt{2\ln x + \ln y}} = \frac{1}{2(2\ln x + \ln y)}$

Division leads: 2

Problem 10

Consider the function $f(x,y) = x^3 - 2xy + y^3$.

Which classification is true for the point $(0,0)$:

A. local minimum B. local maximum C. saddle point D. miscellaneous (i.e. the 2nd order test is not conclusive)

CORRECT ANSWER: C

$f'_x = 3x^2 - 2y \quad f'_y = -2x + 3y^2$

$A = f''_{xx} = 6x,\ B = f''_{xy} = -2,\ C = f''_{yy} = 6y$

Hence, $AC - B^2 = 0 - (-2)^2 = -4 < 0$

So the point (0,0) is a saddle point.

Part III Calculation Problems

5 points per problem

Fractions, logarithms and powers of *e* do not have to be (numerically) calculated.

Problem 11

Compute the value of *x* for which $\frac{7}{x+7} - \frac{5}{x^2-49} = \frac{6}{x-7}$.

ANSWER:

Multiplying the numerator and denominator of the first quotient with (x-7) and multiplying the numerator and denominator of the third quotient with (x+7) yields

$$\frac{7(x-7)}{(x+7)(x-7)} - \frac{5}{x^2-49} = \frac{6(x+7)}{(x-7)(x+7)}$$

$7(x-7) - 5 = 6(x+7) \Leftrightarrow 7x - 49 - 5 = 6x + 42 \Leftrightarrow x = 54+42=96$

Problem 12

Compute $\lim_{x\to 3} \frac{x^2+x-12}{x-3}$.

ANSWER:

$$\lim_{x\to 3} \frac{x^2+x-12}{x-3} = \lim_{x\to 3} \frac{(x+4)(x-3)}{x-3} = \lim_{x\to 3} x + 4 = 7$$

Problem 13

Compute $g''(1)$ if $g(t) = e^{2t^2 + \ln t}$ (the second derivative of g in $t = 1$).

ANSWER:

$g'(t) = (4t + \frac{1}{t})e^{2t^2 + \ln t}$

$g''(t) = (4 - \frac{1}{t^2})e^{2t^2 + \ln t} + (4t + \frac{1}{t})^2 e^{2t^2 + \ln t}$

$g''(1) = 3e^2 + 25e^2 = 28e^2$

Note that $e^{2t^2 + \ln t} = e^{2t^2} e^{\ln t} = te^{2t^2}$.

Problem 14

Find the maximum and minimum of the function $f(x) = \frac{x^2 + 1}{x}$ on the interval $[\frac{1}{2}, 2]$.

ANSWER:

$f(x) = \frac{x^2 + 1}{x} = x + \frac{1}{x}$

$f'(x) = 1 - \frac{1}{x^2}$

$f'(x) = 0$ for $x = 1$ and $x = -1$ (the latter value is outside of the interval).

Function values: $f(½) = (¼ + 1)/½ = 5/2$, $f(1) = (1 + 1)/1 = 2$ and $f(2) = (4 + 1)/2 = 5/2$

So the minimum is 2 and the maximum 5/2.

Problem 15

Consider the function $f(x, y, z) = xy + yz + xz$, with $x = r + s, y = r - s$, and $z = rs$. Calculate the first order derivative of f with respect to r as well as the first order derivative of f with respect to s at the points $r = 1$ and $s = 1$.

ANSWER:

The chain rule leads

$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial r} = (y + z)1 + (x + z)1 + (y + x)s = 1 + 3 + 2 = 6$

$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial s} = (y + z)1 + (x + z)(-1) + (y + x)r = 1 - 3 + 2 = 0$

Problem 16

Consider the function $f(x, y) = 2x^2 + 3y^2$ subject to $y - x = 5$.

Find the stationary points (there could also be none or one) of the Lagrangian function $\mathcal{L}(x, y)$ and the corresponding values of λ.

ANSWER:

The Lagrangian is $\mathcal{L} = 2x^2 + 3y^2 - \lambda(y - x - 5)$

The first order conditions are

$\mathcal{L}'_x = 4x + \lambda = 0 \Leftrightarrow \lambda = -4x$

$\mathcal{L}'_y = 6y - \lambda = 0 \Leftrightarrow \lambda = 6y$

$y - x - 5 = 0$

$\Rightarrow -4x = 6y \Leftrightarrow y = -\frac{2}{3}x$

$\Rightarrow -\frac{2}{3}x - x - 5 = 0 \Leftrightarrow x = -3 \Rightarrow y = 2$ and $\lambda = -4x = 12$

The only stationary point of the Lagrangian is $(-3, 2)$.

Part IV Open Problems

8 points per problem

Fractions, logarithms and powers of *e* do not have to be (numerically) calculated.

Problem 17

Calculate all stationary points for the function $f(x) = x^5 - 5x^4 + 5x^3 + 3$. Determine for each stationary point whether it is a local maximum, a local minimum or an inflection point.

ANSWER:

The stationary points satisfy $f'(x) = 0$. First, $f'(x) = 5x^4 - 20x^3 + 15x^2 = 5x^2(x^2 - 4x + 3) = 5x^2(x-1)(x-3)$. So the stationary points are $x = 0$, $x = 1$ and $x = 3$. Using $f''(x) = 20x^3 - 60x^2 + 30x = 10x(2x^2 - 6x + 3)$ it follows from the second derivative test that $f''(0) = 0$, $f''(1) = 10\cdot(-1) < 0$ and $f''(3) = 30\cdot(3) > 0$. Hence, x=1 corresponds a local maximum and x=3 to a local minimum. To see what happens at x=0, we calculate the roots of $f''(x)$, which turn out to be $0, \frac{3}{2} - \frac{1}{2}\sqrt{3}, \frac{3}{2} + \frac{1}{2}\sqrt{3}$. Note that the last 2 values are strictly positive. One can easily check that for negative values of x the sign of $f''(x)$ is negative, whereas for positive values less than $\frac{3}{2} - \frac{1}{2}\sqrt{3}$, for instance take $x = \frac{1}{2}$, the sign is positive. This means that $f''(x)$ changes sign at x=0 and that 0 is an inflection point. (Alternatively, we could have come to this conclusion by studying the sign diagram of the $f'(x)$. Clearly, for values of x less than 1, the first derivative is always nonnegative. This means that $f'(x)$ does not change sign at x=0, so we don't have alocal extreme point, but an inflection point.)

Problem 18

Given the function $f(x,y) = x^2y^2 - 4y^2 - 9x^2 + 36$.

(a) Calculate all first order partial derivatives.

(b) Calculate all second order partial derivatives.

(c) Find all stationary points of f.

(d) Calculate the value of f in every stationary point and use the second-derivative test to classify them (local minimum, local maximum, saddle point, miscellaneous (i.e. the 2nd order test is not conclusive)).

ANSWER:

(a) $f'_x = 2xy^2 - 18x, \quad f'_y = 2x^2y - 8y$

(b) $A = f''_{xx} = 2y^2 - 18, \quad B = f''_{xy} = f''_{yx} = 4xy, \quad C = f''_{yy} = 2x^2 - 8$

(c) $f'_x = 0 \Rightarrow 2x(y^2-9) = 0 \Leftrightarrow x = 0$ or $y^2 = 9 \Leftrightarrow y = \pm 3$

Using $x = 0$ in $f'_y = 0 \Rightarrow 2y(x^2-4) = 0 \Leftrightarrow y = 0$

Using $y = \pm 3$ in $f'_y = 0 \Rightarrow 2(\pm 3)(x^2-4) = 0 \Leftrightarrow x^2 - 4 = 0 \Leftrightarrow x = \pm 2$

Hence the stationary points are $(0,0), (2,3), (2,-3), (-2,3), (-2,-3)$

(d) For $(0,0)$ we have $A < 0$ and $A \cdot C - B^2 > 0$. Hence, this is a local maximum and the corresponding function value is $f(0,0) = 36$.

For all other stationary points $A \cdot C - B^2 < 0$ and therefore they are saddle points. Furthermore, $f(2,3) = f(2,-3) = f(,2,3) = f(-2,-3) = 0$.

Problem 19

Find all points (there may be only one or none) that correspond to a local maximum of the function $f(x,y) = \frac{1}{3}x^3 + 2y^2$ subject to the constraint $y + x = 3$.

ANSWER:

The Lagrangian function is $\frac{1}{3}x^3 + 2y^2 - \lambda(y + x - 3)$.

The first-order conditions are:

$\mathcal{L}'_x = x^2 - \lambda = 0 \Leftrightarrow \lambda = x^2 \quad (1)$

$\mathcal{L}'_y = 4y - \lambda = 0 \quad (2)$

$x + y = 3 \Leftrightarrow y = 3 - x \quad (3)$

Substituting for $y = 3 - x$ and $\lambda = x^2$ in (2) leads

$4(3-x) - x^2 = 0 \Leftrightarrow -x^2 - 4x + 12 = 0 \overset{Quadratic formula}{\Leftrightarrow} x = \frac{4 \pm \sqrt{16+48}}{-2}$

$\Leftrightarrow x = -6$ or $x = 2$

$\Rightarrow y = 9$ or $y = 1$

So the stationary points are (-6, 9) with $\lambda = 36$ and (2,1) with $\lambda = 4$.

For these points we are going to evaluate

$$D(x,y) = (f''_{11} - \lambda g''_{11})(g'_2)^2 - 2(f''_{12} - \lambda g''_{12}) \cdot g'_1 \cdot g'_2 + (f''_{22} - \lambda g''_{22})(g'_1)^2$$

$$f''_{11} = 2x, f''_{22} = 4, f''_{12} = 0, g'_1 = g'_2 = 1, g''_{11} = g''_{22} = g''_{12} = 0$$

so

$$D(-6,9) = (-12-0)(1) - 2(0-0)(1)(1) + (4-0)(1) = -12 < 0$$

$$D((2,1) = (8-0)(1) - 2(0-0)(1)(1) + (4-0)(1) = 8 > 0$$

This means that (-6,9) is the only local maximum point.