This exam consists of 5 numbered pages.

ERASMUS UNIVERSITY ROTTERDAM

Faculty of Economics
International Bachelor Economics & Business Economics, Bachelor 1

Exam Mathematics 1 (FEB11003X-07)

Wednesday, October 24, 2007, 9:30 – 12:30 hrs

This is a closed book exam. Electronic calculators are not allowed (and not needed).

You are allowed, however, to use a cheat sheet, for which the following rules apply:

- Two-sided A4
- Your name and student number in the right upper corner
- Handwritten in your own handwriting
- No photo copies

All material that does not satisfy these rules will be taken away from you and may be considered a fraud attempt.

Use the separate answer sheet to indicate your answers. The exam consists of four parts, each with a different type of problem. For the basic problems, multiple choice problems and calculation problems you will score 3, 4 and 5 points, respectively, per correct answer and no points for incorrect answers. For the open problems (8 points per problem) your score will depend on the answer and the calculation. The exam grade is the result from the formula (10 + number of scored points)/10. So 63 scored points results in (10+63)/10= 7.3.

Good Luck!

Part II: Multiple Choice Problems Circle the correct answer.									
Problem 5	A	[B]	C	D	Problem 8	A	B	C	[D]
Problem 6	A	B	C	[D]	Problem 9	A	B	[C]	D
Problem 7	A	B	[C]	D	Problem 10	A	[B]	C	D

Part III: Calculation Problems	
Problem 11	$x = -7$ or $x = 6$
Problem 12	$x = 96$
Problem 13	minimum at $t = 3$ and maximum at $t = 27$
Problem 14	$y = \ln\left(\frac{3-x}{4}\right) - 2$ for $x < 3$
Problem 15	degree 2
Problem 16	$e^t(\ln t + \frac{1}{t})$

Part I: Basic Problems

3 points per problem

Problem 1

Describe the difference between the concepts derivative and elasticity in words.

Solution

The derivative of a function is the instantaneous rate of change of the function, while the elasticity of a function is the ratio between the relative change of the function and the relative change of the variable. So the derivative measures how fast the function value changes, while the elasticity measures the percentage change of the function given a percentage change in the variable.

Problem 2

Let $y = f(x)$ be a function. In the context of differentiation, mention two situations in which it is useful to consider $\ln y$ instead of y.

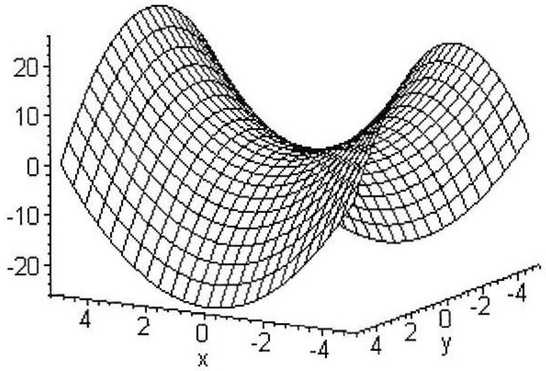
Solution

Taking the logarithm of a function is used in:

- Logarithmic differentiation (see p. 210). Think about differentiating the function $y = x^x$. Taking logarithms gives $\ln y = x \ln x$. Differentiating gives $\frac{y'}{y} = \ln x + 1$ and hence $y' = y(\ln x + 1) = x^x(\ln x + 1)$.
- Calculating elasticities. Instead of using the formula $\text{El}_x y = \frac{x}{y}\frac{dy}{dx}$ one can use the formula $\text{El}_x y = \frac{d \ln y}{d \ln x}$.

Problem 3

Consider the graph of a function with variables x and y in the following figure. Is this function convex, concave or neither of the two?



Solution

The function is neither convex nor concave. The function is concave in the y direction (so for a fixed value of x), while the function is convex in the x direction (so for a fixed value of y).

Problem 4

Let (x_0, y_0) be a stationary point of the function $f(x, y)$ on a convex domain D. Then (x_0, y_0) is a minimum point if for all (x, y) it holds that $f''_{11}(x, y) > 0$, $f''_{22}(x, y) > 0$ and $f''_{11}(x, y)f''_{22}(x, y) - (f''_{12}(x, y))^2 > 0$. Explain why the condition $f''_{22}(x, y) > 0$ does not have to be mentioned explicitly.

Solution

From $f''_{11}(x, y)f''_{22}(x, y) - (f''_{12}(x, y))^2 > 0$ it follows that $f''_{11}(x, y)f''_{22}(x, y) > (f''_{12}(x, y))^2$ and so $f''_{11}(x, y)f''_{22}(x, y) > 0$ as $(f''_{12}(x, y))^2 \geq 0$. Furthermore, because $f''_{11}(x, y) > 0$ we must have $f''_{22}(x, y) > 0$ and hence this condition does not have to be stated explicitly.

Part II: Multiple Choice Problems

4 points per problem

Problem 5

Consider the graph of the function $y = f(x)$. If we replace the graph of $y = f(x)$ by the graph of $y = f(x - c)$ with $c > 0$, then the graph of $f(x)$ shifts

A. c units to the left
B. c units to the right
C. c units upwards
D. c units downwards

Solution

The graph shifts c units to the right.

Problem 6

Let $f(x,y) = \sqrt{\ln(x^2) - \ln(y^2)}$. Calculate the partial elasticity of f w.r.t. x, calculate the partial elasticity of f w.r.t. y, and add them up. Which of the following is the answer?

A. -1 B. $\frac{1}{\ln x - \ln y}$ C. $\frac{2}{\ln x - \ln y}$ D. 0

Solution

We will use the formula $\text{El}_x f = \frac{\partial \ln f}{\partial \ln x}$ and $\text{El}_y f = \frac{\partial \ln f}{\partial \ln y}$. First note that $\ln(x^2) = 2\ln x$ and $\ln(y^2) = 2\ln y$ so that $f(x,y) = \sqrt{2\ln x - 2\ln y}$. Furthermore, $\ln f = \ln\left((2\ln x - 2\ln y)^{\frac{1}{2}}\right) = \frac{1}{2}\ln(2\ln x - 2\ln y)$. It follows that $\text{El}_x f = \frac{\partial \ln f}{\partial \ln x} = \frac{1}{2}\frac{2}{2\ln x - 2\ln y}$. Similarly, $\text{El}_y f = \frac{\partial \ln f}{\partial \ln y} = \frac{1}{2}\frac{-2}{2\ln x - 2\ln y}$. This means that $\text{El}_x f + \text{El}_y f = 0$.

Problem 7

Let $f(x,y) = x^3 + 2xy + \frac{1}{4}y^4 - \ln y$. Find the values of the following derivatives in the point $(1,1)$ and add them up: f'_x, f'_y, f''_{xx}, f''_{xy}, f''_{yx}, and f''_{yy}. Which of the following answers is correct?

A. 18 B. 19 C. 21 D. 22

Solution

We have that $f'_x = 3x^2 + 2y$, $f'_y = 2x + y^3 - \frac{1}{y}$, $f''_{xx} = 6x$, $f''_{xy} = f''_{yx} = 2$, and $f''_{yy} = 3y^2 + \frac{1}{y^2}$. In the point $(1,1)$ it holds $f'_x(1,1) = 5$, $f'_y(1,1) = 2$, $f''_{xx}(1,1) = 6$, $f''_{xy}(1,1) = f''_{yx}(1,1) = 2$, and $f''_{yy}(1,1) = 4$. The sum of the terms is 21.

Problem 8

Determine the first derivative of y (y') in the point $(3,2)$ of $2x^2 + xy - y^2 = -4$ by implicit differentiation. Which of the following answers is correct?

A. 2 B. $\frac{10}{3}$ C. $\frac{17}{4}$ D. 14

Solution

Implicit differentiation gives $4x + xy' + y - 2yy' = 0 \Leftrightarrow y'(x - 2y) = -(4x + y) \Leftrightarrow y' = -\frac{4x+y}{x-2y}$. In the point $(3,2)$ we have $y' = -\frac{4\cdot 3+2}{3-2\cdot 2} = 14$.

Problem 9

Let $f(K, L) = K^2 + L^2 - 2KL$ with $K = \ln(t + 1)$ and $L = e^{3t}$. Calculate the derivative of $f(K, L)$ w.r.t. t for $t = 0$. Which of the following answers is correct?

A. 0 B. 2 C. 4 D. 6

Solution

First, $\frac{df}{dt} = \frac{\partial f}{\partial K}\frac{\partial K}{\partial t} + \frac{\partial f}{\partial L}\frac{\partial L}{\partial t} = (2K - 2L)\frac{1}{t+1} + (2L - 2K)3e^{3t}$. For $t = 0$ we have $K = \ln(0 + 1) = 0$ and $L = e^{3\cdot 0} = 1$. Therefore, for $t = 0$ it holds that $\frac{df}{dt} = (2 \cdot 0 - 2 \cdot 1) \cdot \frac{1}{1+0} + (2 \cdot 1 - 2 \cdot 0) \cdot 3e^{3\cdot 0} = 4$.

Problem 10

Solve $e^{1-t^2} = 4$. Which of the following answers is correct?

A. $t = \pm\sqrt{1 + \ln 4}$ B. $t = \pm\sqrt{1 - \ln 4}$ C. $t = 1 \pm \sqrt{\ln 4}$
D. $t = -1 \pm \sqrt{\ln 4}$

Solution

We have $e^{1-t^2} = 4 \Leftrightarrow 1 - t^2 = \ln 4 \Leftrightarrow t^2 = 1 - \ln 4 \Leftrightarrow t = \pm\sqrt{1 - \ln 4}$.

Part III: Calculation Problems

5 points per problem

Problem 11

Compute the value(s) of x that satisfy the following equation: $5^{x^2+28x-63} = 125^{9x-7}$.

Solution

Because $125 = 5^3$ we have that $125^{9x-7} = (5^3)^{9x-7} = 5^{27x-21}$. So $5^{x^2+28x-63} = 125^{9x-7} \Leftrightarrow 5^{x^2+28x-63} = 5^{27x-21}$. This means that $x^2 + 28x - 63 = 27x - 21 \Leftrightarrow x^2 + x - 42 = 0 \Leftrightarrow (x + 7)(x - 6) = 0 \Leftrightarrow x = -7$ or $x = 6$.

Problem 12

Solve for x: $\frac{7}{x+7} - \frac{5}{x^2-49} = \frac{6}{x-7}$.

Solution

Because $x^2 - 49 = (x + 7)(x - 7)$, we have $\frac{7}{x+7} - \frac{5}{x^2-49} = \frac{6}{x-7} \Leftrightarrow \frac{7(x-7)}{(x+7)(x-7)} - \frac{5}{(x+7)(x-7)} = \frac{6(x+7)}{(x-7)(x+7)} \Leftrightarrow \frac{7x-49}{(x+7)(x-7)} - \frac{5}{(x+7)(x-7)} = \frac{6x+42}{(x-7)(x+7)} \Leftrightarrow 7x - 49 - 5 = 6x + 42 \Leftrightarrow x = 96$.

Problem 13

Find the maximum and minimum point of the function $f(t) = \frac{27}{t} + 3t + 7$ on the interval $[1, 27]$.

Solution

First, we find the stationary points. $f'(t) = -\frac{27}{t^2} + 3 = 0 \Leftrightarrow 3t^2 = 27 \Leftrightarrow t = \pm 3$. Note that $t = -3$ lies not in the interval. Evaluating the function in the stationary point and end points gives $f(1) = 37$, $f(3) = 25$ and $f(27) = 89$. So we have a minimum point at $t = 3$ and a maximum point at $t = 27$.

Problem 14

Find the inverse and its domain of the function $y = 3 - 4e^{x+2}$.

Solution

We have $y = 3 - 4e^{x+2} \Leftrightarrow 3 - y = 4e^{x+2} \Leftrightarrow \frac{3-y}{4} = e^{x+2} \Leftrightarrow \ln\left(\frac{3-y}{4}\right) = x + 2 \Leftrightarrow x = \ln\left(\frac{3-y}{4}\right) - 2$. So the inverse equals $y = \ln\left(\frac{3-x}{4}\right) - 2$ and it is defined for $\frac{3-x}{4} > 0 \Leftrightarrow x < 3$.

Problem 15

Compute the degree of homogeneity of the function $F(x, y, z) = \left(\sqrt{x} + \frac{2x}{\sqrt{y}} + 2\frac{z}{\sqrt{y}}\right)^4$.

Solution

We have $F(tx, ty, tz) = \left(\sqrt{tx} + \frac{2tx}{\sqrt{ty}} + 2\frac{tz}{\sqrt{ty}}\right)^4 = \left(\sqrt{t}\sqrt{x} + \sqrt{t}\frac{2x}{\sqrt{y}} + \sqrt{t}2\frac{z}{\sqrt{y}}\right)^4 = \left(\sqrt{t}(\sqrt{x} + \frac{2x}{\sqrt{y}} + 2\frac{z}{\sqrt{y}})\right)^4 = t^2\left(\sqrt{x} + \frac{2x}{\sqrt{y}} + 2\frac{z}{\sqrt{y}}\right)^4 = t^2 F(x, y, z)$. This means that the function is homogeneous of degree 2.

Problem 16

Consider the function $F(x, y) = xy$ with $x = e^t$ and $y = \ln t$. Compute the derivative of $F(x, y)$ w.r.t. t.

Solution

We have $\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t} = ye^t + x\frac{1}{t} = \ln t \cdot e^t + e^t\frac{1}{t} = e^t(\ln t + \frac{1}{t})$.

Part IV: Open problems

8 points per problem

Problem 17 ($2 + 2 + 2 + 2$ points)

Consider the function $f(x) = x^6 - 10x^4$.

a. Calculate the first derivative and the stationary point(s).

b. Classify the stationary point(s) using the first derivative (local minimum, local maximum or not an extreme point).

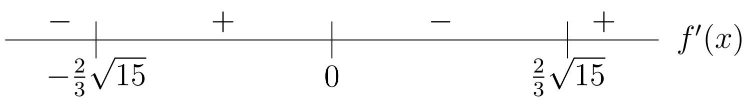
c. Calculate the second derivative $f''(x)$ and the point(s) for which $f''(x) = 0$.

d. Classify the point(s) from part c using the second derivative (inflection point or not an inflection point).

Solution

a. We have $f'(x) = 6x^5 - 40x^3 = 2x^3(3x^2 - 20) = 0 \Leftrightarrow x = 0$ or $3x^2 = 20 \Leftrightarrow x = 0$ or $x = \pm\frac{2}{3}\sqrt{15}$.

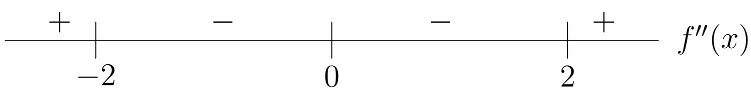
b. We use the sign diagram of the first derivative to classify the points.



It follows that $x = 0$ is a local maximum point and $x = \pm\frac{2}{3}\sqrt{15}$ are local minimum points.

c. Using part a. we have $f''(x) = 30x^4 - 120x^2 = 30x^2(x^2 - 4) = 0 \Leftrightarrow x = 0$ or $x = \pm 2$.

d. We use the sign diagram of the second derivative to classify the points.



It follows that $x = \pm 2$ are inflection points and $x = 0$ is no inflection point (the latter also follows from part b.).

Problem 18 $(2 + 2 + 2 + 2$ points)

Given the function $f(x, y) = x^3 - 3xy + y^3$.

a. Calculate the first order partial derivatives of f.

b. Calculate the second order partial derivatives of f.

c. Calculate the stationary points of f.

d. Calculate the function value of f for every stationary point and use the second order test to classify them (local minimum, local maximum, saddle point), if possible.

Solution

a. $f'_x = 3x^2 - 3y$, $f'_y = -3x + 3y^2$

b. $f''_{xx} = 6x$, $f''_{xy} = -3$, $f''_{yy} = 6y$

c. For the stationary points it holds that $f'_x = 0$ and $f'_y = 0$. From $f'_x = 3x^2 - 3y = 0$ it follows that $y = x^2$. Substituting this in $f'_y = -3x + 3y^2 = 0$ gives $-3x + 3(x^2)^2 = 0 \Leftrightarrow x(1 - x^3) = 0 \Leftrightarrow x = 0$ or $x = 1$. So the stationary points are $(0, 0)$ and $(1, 1)$.

d. Using the second order test we get the following table.

(x, y)	A	B	C	$AC - B^2$	type
$(0, 0)$	0	-3	0	-9	saddle point
$(1, 1)$	6	-3	6	27	local minimum

We have a saddle point at $(0, 0)$ and a local minimum point at $(1, 1)$.

Problem 19 $(2+2+2+2$ points)

Assume we want to maximize or minimize the function $f(x,y) = 3x + y$ subject to the constraint $g(x,y) = x^3y = 81$.

a. Give the Lagrangian function.

b. Give the first order conditions of the Lagrange multiplier rule.

c. Calculate the point(s) that satisfy the first order conditions.

d. What is the economic interpretation of the value of λ?

Solution

a. $\mathcal{L}(x,y) = f(x,y) - \lambda(g(x,y) - c) = 3x + y - \lambda(x^3y - 81)$.

b. In general the first order conditions are:

$$\mathcal{L}_1'(x,y) = f_1'(x,y) - \lambda g_1'(x,y) = 0$$
$$\mathcal{L}_2'(x,y) = f_2'(x,y) - \lambda g_2'(x,y) = 0$$
$$g(x,y) = c.$$

For this specific problem it gives:

$$\mathcal{L}_1'(x,y) = 3 - 3\lambda x^2 y = 0 \tag{1}$$
$$\mathcal{L}_2'(x,y) = 1 - \lambda x^3 = 0 \tag{2}$$
$$x^3 y = 81. \tag{3}$$

c. From (2) it follows that $\lambda = \frac{1}{x^3}$ and from (3) it follows that $y = \frac{81}{x^3}$. Substituting this into (1) gives $3 - 3\frac{1}{x^3}x^2\frac{81}{x^3} = 0 \Leftrightarrow 1 - \frac{81}{x^4} = 0 \Leftrightarrow x^4 = 81 \Leftrightarrow x = \pm 3$. For $x = 3$ we have $y = 3$ and $\lambda = \frac{1}{27}$ and for $x = -3$ we have $y = 3$ and $\lambda = -\frac{1}{27}$. So we have the stationary point $(3,3)$ with $\lambda = \frac{1}{27}$ and the stationary point $(-3,-3)$ with $\lambda = -\frac{1}{27}$.

d. The value of λ gives the rate of change of the optimal value w.r.t. the value c, so it is the derivative of the optimal value w.r.t. c.