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Simplify:

$x^a x^b = x^{a+b}$

$(x^a)^b = x^{ab}$

$x^a y^a = (xy)^a$

$\sqrt{x} = x^{1/2}$

$\sqrt{xy} = \sqrt{x} * \sqrt{y}$

$x = \log_a y \rightarrow a^x = y$

$\log_e x = \ln x$

$x = \ln y \rightarrow e^x = y$

$a^x = b \rightarrow x = \dfrac{\ln b}{\ln a}$

$\ln ab = \ln a + \ln b$

$\ln \dfrac{a}{b} = \ln a - \ln b$

$\ln a^b = b \ln a$

Derivatives:

F(x)	$f(x)$	$f'(x)$
$\dfrac{2x^{3/2}}{3}$	$\sqrt{x}$	$\dfrac{1}{2\sqrt{x}}$
e^x	e^x	e^x
$\dfrac{e^{ax}}{a}$	e^{ax}	ae^{ax}
$x(\ln(x)-1)$	$\ln x$	$\dfrac{1}{x}$
$\dfrac{a^x}{\ln(a)}$	a^x	$a^x \ln a$

Integration examples:

$\int f(x)dx = F(x) + C$

$\int x dx = \frac{1}{2}x^2 + C$

$\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + C$

$\int e^x dx = e^x + C$

$\int x^p dx = \frac{1}{p+1}x^{p+1} + C, p \neq -1$

$\int \frac{1}{x} dx = \ln|x| + C$

$\int e^{ax} dx = \frac{1}{a}e^{ax} + C$

$\int a^x dx = \frac{1}{\ln a}a^x + C, a > 0 \,\&\, a \neq 1$

Formula not defined at asked value:

$\int_0^2 \frac{1}{x} dx = \lim_{h \to 0+} \int_h^2 \frac{1}{x} dx$

f(x) is defined [a,b], but not at a and b:

$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Integration general rules:

$\int af(x)dx = a\int f(x)dx$

$\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$

$\int [f(x)-g(x)]dx = \int f(x)dx - \int g(x)dx$

$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$

Area and definite integrals:

$A(b) = F(b) - F(a) = \int_a^b f(x)dx$

For calculation of area that (also) has negative values, calculate x=0's, determine areas individually and add them up, take absolute values.

Properties of definite integrals:

$\int_a^b f(x)dx = -\int_b^a f(x)dx$

$\int_a^a f(x)dx = 0$

$\int_a^b pf(x)dx = p\int_a^b f(x)dx$

$\int_a^b f(x)dx = \int_a^c f(x) + \int_c^b f(x), a < c < b$

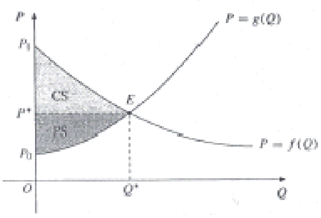
$\int_a^b [pf(x) + rg(x)]dx = p\int_a^b f(x) + r\int_a^b g(x)$

Calculating derivatives of integrals can be done manually, or in this form:

If: $I(x) = \int_{g(x)}^{h(x)} f(t)dt$, then:

$I'(x) = f(h(x)) \cdot h'(x) - f(g(x)) \cdot g'(x)$

Consumer and producer surplus:



$CS = \int_0^{Q*} [f(Q) - P*]dQ$

$PS = \int_0^{Q*} [P* - g(Q)]dQ$

Integration by substitution:

$\int (x^2+10)^{50} 2x dx$

$u = x^2 + 10 \quad du = 2x dx$

$\int u^{50} du = \frac{1}{51}u^{51} + C$

$\int (x^2+10)^{50} 2x dx = \frac{1}{51}(x^2+10)^{51} + C$

General rule: $(u = g(x))$

$f(g(x)) \cdot g'(x)dx = \int f(u)du$

Infinite intervals of integration:

$\int_1^b \frac{1}{x^2} dx = \left[-\frac{1}{x}\right]_1^b = 1 - \frac{1}{b}$

$\int_1^\infty \frac{1}{x^2} dx = \lim_{b \to \infty} \int_1^b \frac{1}{x^2} dx = \lim_{b \to \infty} (1 - \frac{1}{b}) = 1$

The following improper integral:

$\int_a^\infty f(x)dx = \lim_{b \to \infty} \int_a^b f(x)dx$

Is called convergent if the limit exists (as a finite number); then *f* is said to be integrable over [a,∞); otherwise the integral is called divergent.

$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_0^\infty f(x)dx$

$\int_{-\infty}^{\infty} f(x)dx \neq \lim_{b \to \infty} \int_{-b}^{b} f(x)dx$

Interest, computed periodically:

$S_t = S_0(1+r)^t$ (r =interest %/period, S_0 =principal, t = period number)

With periods: $S_t = S_0(1+\frac{r}{n})^{nt}$

When interest is added *n* times per period, the effective periodic rate is:

$R = (1+\frac{r}{n})^n - 1$

Interest, computed continuously:

$S(t) = S_0 \cdot e^{rt}$, effective periodic interest rate: $R = e^r - 1$

Geometric series, finite sums:

$S_n = a + ak + ak^2 + ... + ak^{n-1}$

$S_n = a \cdot \frac{k^n - 1}{k - 1} \quad if: k \neq 1$

Infinite sums:

$\sum_{n=0}^{\infty} a(k)^n = a + ak... = \frac{a}{1-k} \; (|k| < 1)$

Present value of an annuity A:

$P_n = \frac{A}{r}\left(1 - \frac{1}{(1+r)^n}\right)$

$P = \lim_{n \to \infty} P_n = \frac{A}{r}$

Continuous income stream from t0→t =T at the rate of f(t) per year at time t

Present discounted value:

$PDV = \int_0^T f(t)e^{-rt}dt$

Future discounted value:

$FDV = e^{rT}\int_0^T f(t)e^{-rt}dt = e^{rT} \cdot PDV$

Matrices:

(m x n) matrix has *m* rows, *n* columns

Addition and substraction:

Only for same dimensions

$\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 8 & -1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 2 & -4 \\ 0 & 1 \end{bmatrix}$

$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 7 & -2 \\ 8 & 0 \end{bmatrix}$

Scalar multiplication:

$3\mathbf{A} = \begin{bmatrix} 15 & 6 \\ 24 & -3 \end{bmatrix} \quad -\mathbf{A} = \begin{bmatrix} -5 & -2 \\ -8 & 1 \end{bmatrix}$

Matrix multiplication AB:

Only when: A # columns = # rows of B

$\mathbf{AB} = \begin{bmatrix} 10 & -18 \\ 16 & -33 \end{bmatrix} \quad \mathbf{AB} \neq \mathbf{BA}$

a_{ij} : i-th row, j-th column; eg. $a_{12} = 2$

Identity matrix:

$\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\mathbf{I}_m\mathbf{A} = \mathbf{A} \quad \mathbf{AI}_n = \mathbf{A}$

$\mathbf{AB} = \mathbf{0}$ does not imply $\mathbf{A} = \mathbf{0}$ / $\mathbf{B} = \mathbf{0}$

$\mathbf{AB} = \mathbf{AC}$ & $\mathbf{A} \neq \mathbf{0}$ don't imply $\mathbf{B} = \mathbf{C}$

Transpose matrix:

Transpose matrix **A'** of **A**: $a'_{ij} = a_{ji}$

$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 7 & 5 \end{bmatrix}$ then $\mathbf{A}' = \begin{bmatrix} 1 & 0 & 7 \\ 2 & 1 & 5 \end{bmatrix}$

A matrix is symmetric when all non-diagonal values are equal.

Symmetric matrix: $\mathbf{A}' = \mathbf{A}$

Leontief model:

$x_1 = \beta x_2 + d_1$

$x_2 = \gamma x_3 + d_2$

$x_3 = \alpha x_1$, these equations give:

$\begin{bmatrix} 1 & -\beta & 0 \\ 0 & 1 & -\gamma \\ -\alpha & 0 & 1 \end{bmatrix} \cdot x = \begin{bmatrix} d_1 \\ d_2 \\ 0 \end{bmatrix}$

Unique solution only if $1 - \alpha\beta\gamma \neq 0$

$x_1 = \frac{d_1 + \beta d_2}{1 - \alpha\beta\gamma} \quad x_1, x_2, x_3 \geq 0$ only if:

$x_2 = \frac{\alpha\lambda d_1 + d_2}{1 - \alpha\beta\gamma} \quad \alpha\beta\gamma < 1$

$x_3 = \frac{\alpha d_1 + \alpha\beta d_2}{1 - \alpha\beta\gamma}$

Gaussian elimination:

-Make identity matrix by +/-other rows & row multiplication by a constant

-Rows can be interchanged freely

-There is no unique solution when one row has only zeroes. The variable of that row can then be chosen freely; one degree of freedom

-No solution possible if one row of **A** has only zeroes, but vector **B** has value

Vectors:

Column vector: (*n* x 1) matrix

Row vector: (1 x *n*) matrix

Inner product of two vectors:

$\mathbf{a} = (a_1, ..., a_n) \quad \mathbf{b} = (b_1, ..., b_n)$

$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + ...a_nb_n = \sum_{i=1}^{n} a_ib_i$

Vector **c** is a linear combination of vectors **a** and **b** if **c** can be written as: **c**=*t***a**+*s***b** with *t, s* real numbers

Determinants of order 2:

$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

Det of order 3 and cofactors:

$C_{ij} = (-1)^{i+j} \times$ det of matrix after deleting *i*-th row and *j*-th column

$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Expansion in terms of row *i*:

$|\mathbf{A}| = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3}$

Expansion in terms of column *j*:

$|\mathbf{A}| = a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j}$

Rules for determinants:

-If all elements in a row (or column) of **A** are equal to zero, then $|\mathbf{A}| = 0$

- $|\mathbf{A}'| = |\mathbf{A}|$ r/c=row/column

-If all elements in a single r/c are multiplied by a number α, then the determinant is also multiplied by α

-If two r/c are interchanged, the determinant changes sign

-If 2 r/c of **A** are proportional $|\mathbf{A}| = 0$

-Det. is unchanged if a multiple of one r/c is added to a different r/c
-Det. of two $n \times n$ matrixes **A** and **B** is the product: $|\mathbf{AB}| = |\mathbf{A}| \cdot |\mathbf{B}|$
-**A** is a $n \times n$ matrix and α is a real number, then $|\alpha\mathbf{A}| = \alpha^n \cdot |\mathbf{A}|$

Matrix inverses:
$n \times n$ matrix **A** has inverse **X** if:
$\mathbf{AX} = \mathbf{XA} = \mathbf{I}_n$
Only if **A** is <u>nonsingular ($|\mathbf{A}| \neq 0$)</u> it can and must have an inverse $\mathbf{A}^{-1}$

Finding the inverse - row operations:
Construct $n \times 2n$ matrix (**A**|**I**), apply row operations to get (**I**|**B**), $\mathbf{B} = \mathbf{A}^{-1}$

Finding the inverse – using cofactors:
$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$
Matrix on right is called adjoint of **A**, denoted as adj(**A**). Mind the r/c's!

Cramer's Rule:
A is a square matrix and nonsingular.
$$\alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3 = b_1$$
$$\alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{23}x_3 = b_2$$
$$\alpha_{31}x_1 + \alpha_{32}x_2 + \alpha_{33}x_3 = b_3$$
D_j is det. of matrix where the *j*-th column of **A** has been replaced by vector **B.**
$$x_1 = \frac{D_1}{|\mathbf{A}|} \quad x_j = \frac{D_j}{|\mathbf{A}|}$$

Market equilibrium:
S_t = supply in period *t*
D_t = demand P_t = price
$$D_t = \alpha - \beta P_t$$
$$S_t = -\gamma + \delta P_{t-1}$$
<u>Solution</u>: $P_t = bP_{t-1} + c$
where $b = -\delta/\beta$ & $c = (\alpha+\gamma)/\beta$

1st order linear difference equations:
<u>Model</u>: $y_{t+1} = by_t + c$
Or: $y_{t+1} - y_t = (b-1)y_t + c$
General (total) solution:
$$y_t = Ab^t + \frac{c}{1-b} \quad (b \neq 1)$$
$$y_t = A + ct \quad (b \equiv 1)$$
If y_0 is known, then:
$$A = y_0 - \frac{c}{1-b} \quad (b \neq 1)$$
$$A = y_0 \quad (b \equiv 1)$$

Linear right-hand-side coefficients:
<u>Model: $y_{t+1} - 3y_t = 2t$</u>
Complementary solution: $A3^t$
try $Bt + C$ for y_t, then:
$$y_{t+1} - 3y_t = B(t+1) + C - 3Bt =$$
$$-2Bt + (B - 2C) = 2t \text{, so:}$$
$$-2Bt = 2t \ \& \ B - 2C = 0$$
$$\Rightarrow B = -1,\ C = -\tfrac{1}{2}$$
Solution: $y_t = A3^t - t - \tfrac{1}{2}$

Non-linear right-hand-side:
<u>Model</u>: $y_{t+1} = -3y_t = t^2$
Complementary solution: $A3^t$
Try $Bt^2 + Ct + D$ for y_t, then:
$$B(t+1)^2 + C(t+1) + D - 3Bt^2 - 3Ct - 3D =$$
$$-2Bt^2 + (2B - 2C)t + (B + C - 2D)$$
Which has to equal $1t^2 + 0t + 0$, so:
$$-2B = 1,\ 2B - 2C = 0,\ B + C - 2D = 0$$
$$\Rightarrow B = \tfrac{1}{2},\ C = -\tfrac{1}{2},\ D = -\tfrac{1}{2}$$
<u>Solution</u>: $y_t = A3^t - \tfrac{1}{2}t^2 - \tfrac{1}{2}t - \tfrac{1}{2}$

Dynamic stability:
Consider $y_{t+1} = by_t + c$
Does y_t converge when $t \to \infty$?
$b = 1: y_t = A + ct$ no convergence (unless c=0), so unstable
$b \neq 1: y_t = Ab^t + \frac{c}{1-b}$
Determining factor is b^t:
$|b| < 1$ convergence to $\frac{c}{1-b}$; stable
$|b| > 1$ no convergence (±∞); unstable
$b = -1$ uniform oscillation ±1 unstable

Cobweb model:
$$D_t = \alpha - \beta P_t$$
$$S_t = -\gamma + \delta P_{t-1}$$
$$\alpha, \beta, \gamma, \delta > 0$$
$$P_t = \left(P_0 - \frac{\alpha+\gamma}{\beta+\delta}\right)\left(-\frac{\delta}{\beta}\right)^t + \frac{\alpha+\gamma}{\beta+\delta}$$
$\delta < \beta$: stable $\delta > \beta$: unstable
$\delta = \beta$: unstable; uniform oscillation
Time path (convergent or divergent, monotone or oscillating):
$\delta > \beta$: divergent, oscillating, unstable
$\delta < \beta$: convergent, oscillating, stable

ABC-formula:
$$ax^2 + bx + c = 0$$
$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Examples:
<u>Integration examples:</u>
$$\int xe^{-cx^2}dx$$
$$u = -cx^2 \quad du = -2cxdx$$
so $xdx = -\frac{1}{2c}du$
$$\int xe^{-cx^2}dx = -\frac{1}{2c}\int e^u du =$$
$$-\frac{1}{2c}e^u + C = -\frac{1}{2c}e^{-cx^2} + C$$

$$\int_1^e \frac{1+\ln x}{x}dx$$
$$u = 1 + \ln x \quad du = \frac{1}{x}dx$$
$$\int u du = \frac{1}{2}u^2 + C = \frac{1}{2}(1+\ln x)^2 + C$$
$$\frac{1}{2}\left[(1+\ln x)^2\right]_1^e = \frac{1}{2}\left((1+1)^2 - (1+0)^2\right) = \frac{3}{2}$$

Present value of 15 annual deposits of \$3500; first deposit after one year, annual interest 12%
$n = 15 \quad A = 3500 \quad r = 0.12$
$$P_{15} = \frac{3500}{0.15} \cdot \left(1 - \frac{1}{1.12^{15}}\right) = 23838$$

Calculate $\int_1^2 \frac{e^x + e^{-x}}{e^x - e^{-x}}dx$
$$u = e^x - e^x \quad du = (e^x + e^{-x})dx$$
$$\int \frac{1}{u}du = \ln|u| = \ln\left|e^x - e^{-x}\right| + C$$
$$\int_1^2 \frac{e^x + e^{-x}}{e^x - e^{-x}}dx = \left[\ln\left|e^x - e^{-x}\right|\right]_1^2 =$$
$$\ln(e^2 - e^{-2}) - \ln(e^1 - e^{-1}) =$$
$$\ln\left(\frac{e^2 - e^{-2}}{e^1 - e^{-1}}\right) = \ln\left(\frac{(e^1 - e^{-1})(e^1 + e^{-1})}{e^1 - e^{-1}}\right)$$
$$= \ln(e + e^{-1})$$

Compute $\sum_{k=2}^{\infty} 6\left(\frac{2}{3}\right)^k$
$$6 \cdot \frac{4}{9} + 6 \cdot \frac{8}{27} + 6 \cdot \frac{16}{81} + \ldots =$$
$$\frac{8}{3} + \frac{8}{3}\left(\frac{2}{3}\right) + \frac{8}{3}\left(\frac{2}{3}\right)^2$$
Hence $a = \frac{8}{3}; k = \frac{2}{3}$
$$\sum_{k=2}^{\infty} 6\left(\frac{2}{3}\right)^k = \frac{\frac{8}{3}}{1 - \frac{2}{3}} = 8$$

PDV calculations:
<u>Constant income stream of \$1800 per year for next 3 years, interest = 9%/y:</u>
$$PDV = \int_0^3 1800e^{-0.09t}dt = \left[1800\left(-\frac{e^{-0.09t}}{0.09}\right)\right]_0^3$$
$$= \frac{1800}{0.09}\left(1 - e^{-0.27}\right) = 20000(1 - e^{-0.27})$$
<u>Constant income stream of \$800 per year for next 8 years, interest=4%/y:</u>
$$PDV = \int_0^8 800e^{-0.04t}dt = \left[800\left(-\frac{e^{-0.04t}}{0.04}\right)\right]_0^8$$
$$= \frac{800}{0.04}\left(1 - e^{-0.32}\right) = 20000(1 - e^{-0.32})$$
<u>Constant income stream of \$1400 per year for next 4 years, interest=7%/y:</u>
$$PDV = \int_0^4 1400e^{-0.07t}dt = \left[1400\left(-\frac{e^{-0.07t}}{0.07}\right)\right]_0^4$$
$$= \frac{1400}{0.07}\left(1 - e^{-0.28}\right) = 20000(1 - e^{-0.28})$$

Compute $\int_1^2 f(x)dx$, where $f(2) = 22$ and $f'(x) = 12x + \frac{6}{x^2}$ Solution:
$$f(x) = 6x^2 - \frac{6}{x} + c$$
$$f(2) = 22 = 24 - 3 + c \Rightarrow c = 1$$
$$\int_1^2 f(x)dx = \left[2x^3 - 6\ln x + x\right]_1^2 =$$
$$16 - 6\ln 2 + 2 - (2 - 0 + 1) = 15 - 6\ln 2$$

Find the following indefinite integral:
$$\int \frac{2x}{6x^2 + 7}dx$$
$$u = 6x^2 + 7 \quad du = 12xdx$$
$$\int \frac{2x}{6x^2 + 7}dx = \int \frac{2x}{u}\frac{du}{12x} = \int \frac{1}{6u}du =$$
$$\frac{1}{6}\ln|u| + C = \frac{1}{6}\ln(6x^2 + 7) + C$$

$$\int \frac{x^2 - x}{x^3}dx = \int \frac{x-1}{x^2}dx =$$
$$\int\left(\frac{1}{x} - \frac{1}{x^2}\right)dx = \ln|x| + \frac{1}{x} + C$$

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$
$$\mathbf{AB} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} - \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 2a + c & 2b \\ -a & -b \end{pmatrix} - \begin{pmatrix} 2b & a + b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 2a + c - 2b & b - a \\ -a & -b - c \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ 0 & 1 \end{pmatrix}$$
$$\Rightarrow a = 0 \quad b = -2 \quad c = 1$$